Index of Lie algebras and *Meanders*

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Common definitions

- *Lie algebra* is a vector space with a bracket $[\cdot,\cdot]$ operation satisfying Jacobi identity. An example is the Lie algebra $\mathfrak{gl}(n)$ of all $n \times n$ matrices, $[A,B] = AB - BA$.

- The *kernel* of a matrix A is the set of all vector solutions of $A\vec{v} = 0$. If A is a skew-symmetric matrix we will call the kernel dimension of A *index* of A.

- The dual space of linear functions on Lie algebra $\mathfrak{g}$ is denoted $\mathfrak{g}^*$ and called coadjoint space.

Index of Lie algebras

The bracket operation on Lie algebras gives rise to a skew-symmetric form. Let f be a linear function on Lie algebra $\mathfrak{g}$. Then

$$B_f(X,Y) := f([X,Y])$$

is skew-symmetric in X and Y.

Index of B_f varies with f. For example when $f = 0$ it is equal to $\dim \mathfrak{g}$.

By definition *index* of Lie algebra $\mathfrak{g}$ is

$$\mathrm{I}(\mathfrak{g}) = \min_{f \in \mathfrak{g}^*} \text{index } B_f$$

Properties of Index

- The field of invariants of coadjoint action on $\mathfrak{g}^*$ is generated by $\mathrm{I}(\mathfrak{g})$ rational functions.

- The set of f such that index of B_f is equal to $\mathrm{I}(\mathfrak{g})$ is open in $\mathfrak{g}^*$.

- In any Lie algebra $\mathfrak{g}$ there exists a commutative subalgebra of dimension $\mathrm{I}(\mathfrak{g})$.

Coadjoint action

Coadjoint action is yet another way to look at the form B_f. If we fix the first argument the result is a function $B_f(X,\cdot)$ on $\mathfrak{g}$. By definition:

$$\mathrm{ad}^*_X : f \to B_f(-X,\cdot)$$

ad^*_X is a linear operator on $\mathfrak{g}^*$. The property $[\mathrm{ad}^*_X, \mathrm{ad}^*_Y] = \mathrm{ad}^*_{[X,Y]}$ follows from Jacobi identity and means that ad^* is a representation of Lie algebra $\mathfrak{g}$ on vector space $\mathfrak{g}^*$.

$\mathrm{I}(\mathfrak{g})$: How to Compute ?

- Pick an $f \in \mathfrak{g}^*$, compute $\dim \ker B_f$.
  Pros: easy to compute for special f, gives estimate from above.
  Cons: hard to do for general f or to prove that a given f is precise.

- Dimensional consideration: $\dim \mathfrak{g} - \mathrm{I}(\mathfrak{g})$ should be even.
  Pros: relatively straightforward, estimate from below.
  Cons: does not work well when $\mathrm{I}(\mathfrak{g})$ is large.

- Wedge products: consider B_f as a 2-form Ω with coefficients depending on f. Then the highest power r when $\wedge^r\Omega$ is non-zero gives $\frac{1}{2}(\dim \mathfrak{g} - \mathrm{I}(\mathfrak{g}))$.
  Pros: Straightforward, gives precise answer.
  Cons: Easy to compute only for $\mathfrak{g}$ of small dimension.

$\mathfrak{gl}(N)$

$\mathfrak{gl}(N)$ is the algebra of $N \times N$ matrices. The coadjoint space can be realized as $N \times N$ matrices:

$$F(X) = \mathrm{tr}(FX)$$

$$\begin{aligned} B_F(X,Y) &= \mathrm{tr}(F[X,Y]) = \mathrm{tr}(FXY - FYX) \\ &= \mathrm{tr}(FXY - XFY) = \mathrm{tr}([F,X]Y) \end{aligned}$$

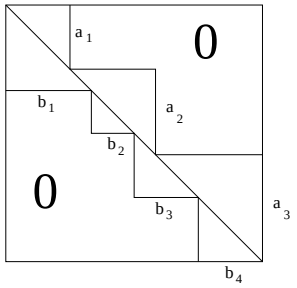
Thus $X \in \ker B_F$ if and only if $[F,X] = 0$. For F with different eigenvalues $\dim \ker B_F = N$.

Since Ω has polynomial coefficients and because $\{F : F$ *has different eigenvalues*$\}$ is open

$$\mathrm{I}(\mathfrak{gl}(N)) = N$$

Seaweed algebras

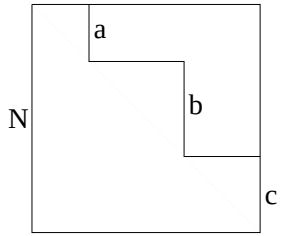
Elements of $\mathfrak{gl}(N)$ are often decomposed into products of upper- and lower-triangular matrices. *Seaweed algebras* are formed by products of triangular matrices of different sizes.



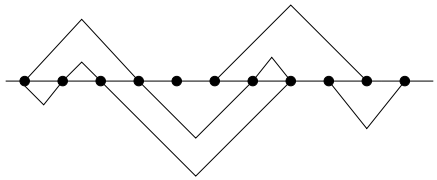
On the above picture the shape is fixed. All matrices of this shape form an associative algebra. It is also a Lie algebra with the bracket $[A,B] = AB - BA$.

Seaweed algebras

The sizes of triangles above the diagonal and below it form two ordered partitions of number N - size of the whole matrix. A special case when one of the partitions is (N) is known in literature as *parabolic subalgebras of* $\mathfrak{gl}(N)$



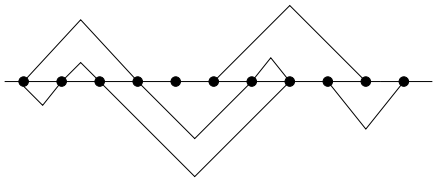
Meanders



In general, meanders are equivalence classes of circle (and/or segment) embeddings into plane. It is convenient for us to allow *meander* to consist of several connected components.

Each arc defines a transposition of points on the straight line. The product of these transpositions over all arcs is a permutation associated to a given meander (it is not unique).

Meanders



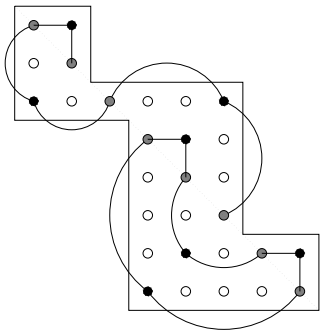
Though the associated permutation is not unique the number of cycles in its decomposition is.

It is equal to twice the number of cycles in the *meander* plus the number of chains plus the number of isolated points.

For the *meander* above:

$$[(1,4)(2,3)(6,10)(7,8)] \cdot$$
$$[(1,2)(3,8)(4,7)(9,11)] =$$
$$(1,3,7)(2,4,8)(5)(6,10)(9,11)$$

Meander associated to Seaweed algebra



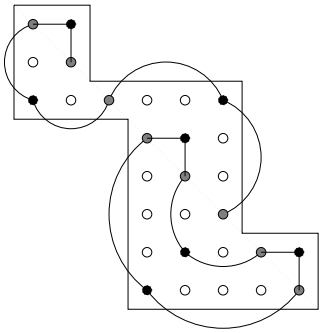
Each *black* element is connected by an edge to the *grey* elements in the same column or row.

Answer

The index of seaweed algebra is equal to:

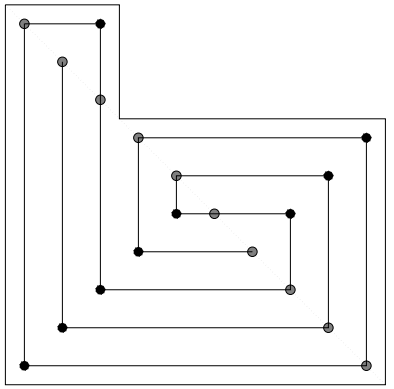
- twice the number of cycles in the associated meander plus the number of chains plus the number of isolated vertices.
- the number of cycles in associated permutation.

Example: $(2,4,2) : (3,5)$



$[(12)][(36)(45)][(78)] \cdot [(13)][(48)(57)]$
$= (1632)(47)(58)$ (right multiplication)

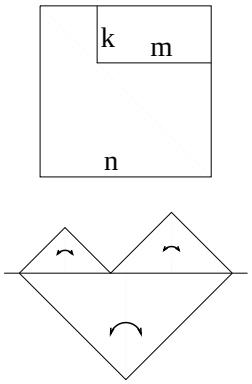
Example: $(3,7) : (10)$



$[(1,3)][(4,10)(5,9)(6,8)] \cdot$
$[(1,10)(2,9)(3,8)(4,7)(5,6)] =$
$= (1,4,7,10,3,6,9,2,5,8)$

Maximal parabolic

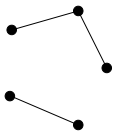
Though in general elementary formula is unknown, one can derive a simple answer for the maximal parabolic case: the number of cycles in the associated permutation is $\gcd(k,m)$.



Remarks

The proof is based on the following simple fact:

Let Ω be a skew-symmetric form (e.g. $\Omega = \xi_1 \wedge \xi_2 + 2\xi_2 \wedge \xi_3 + \xi_4 \wedge \xi_5$). Make a graph with vertices labeled by elements of the basis (ξ_i). Connect two vertices ξ_i and ξ_j by an edge only if the monomial $\xi_i \wedge \xi_j$ has a non-zero coefficient in Ω:



Then

$$\max_{\wedge^r \Omega \neq 0} (r) \leq \max_{S=\{\text{disjoint edge set}\}} (\#S)$$

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