Stat 620, Fall 2005

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Background material from real analysis and topology

1 The Real Line and its Subsets

Notation 1.1 Let $\mathbb{R}$ denote the set of all real numbers and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ be the extended real line. Let $\mathbb{Q}$ be the set of all rationals. Denote by $\emptyset$ an empty set. Also define

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}, \ \ (a, b] = \{x \in \mathbb{R} : a < x \leq b\}, \ \ etc.$$

Definition 1.1 A set $E \subset \mathbb{R}$ is *bounded above* if $\exists M \in \mathbb{R}$ such that $x \leq M \ \ \forall x \in E$. A number $L \in \overline{\mathbb{R}}$ is called a least upper bound or supremum of a given set E (written $L = \sup E$ or $\sup\{x : x \in E\}$) if

(i) $x \leq L$ for all $x \in E$;

(ii) If M is *any* upper bound of E, then $L \leq M$.

After replacing " $\leq$ " sign with " $\geq$ " in the definition above, one arrives at the notions of *boundedness below* and *an upper lower bound* or *infimum* of a set E (written as $\inf E$ or $\inf\{x : x \in E\}$).

If a set is bounded both above and below, it's called *bounded*.

Example 1.1 The set $E = \mathbb{Q} \cap (-\infty, \pi)$ is bounded above with $\sup E = \pi$, but is unbounded below, i.e. $\inf E = -\infty$. Note also that $\pi \notin E$, which illustrates that $\sup E$ need not belong to E.

Proposition 1.1 *If a set E is bounded above then it has a (unique) finite supremum. Similarly, if a set E is bounded below then it has a (unique) finite infimum.*

Proposition 1.2 *(a) Let E be a set bounded above. Then* $\sup E = L$ *iff*

(i) $x \leq L \ \ \forall x \in E$ *and*

(ii) $\forall \epsilon > 0 \ \exists x \in E$ *such that* $x > L - \epsilon$.

(b) Let E be a set bounded below. Then $\inf E = l$ *iff*

(i) $x \geq l \ \ \forall x \in E$ *and*

(ii) $\forall \epsilon > 0 \ \exists x \in E$ *such that* $x < l + \epsilon$.

Proposition 1.3 *Suppose* $\sup E = L \notin E$, *then* $\exists (x_n)_{n \geq 1}$, *an infinite sequence of distinct points in E, such that* $\lim_{n \to \infty} x_n = L$. *Similarly, if* $\inf E = l \notin E$, *then* $\exists (y_n)_{n \geq 1}$, *an infinite sequence of distinct points in E, such that* $\lim_{n \to \infty} y_n = l$.

Theorem 1.1 (Bolzano-Weierstrass.) *Every infinite bounded set* $E \subset \mathbb{R}$ *has at least one limit point* $x \in E$, *i.e.* $\forall \epsilon > 0 \ \exists y \in E$ *s.t.* $y \neq x$ *and* $|y - x| < \epsilon$.

Definition 1.2 Fix an arbitrary sequence $(a_n)_{n\geq 1} \in \mathbb{R}$. For each $n \geq 1$, consider the set $E_n = \{a_n, a_{n+1}, a_{n+2}, \dots\}$. Define the *limit superior* ($\limsup_{n\to\infty}$ or $\overline{\lim}_{n\to\infty}$) of (a_n) by

$$\overline{\lim_{n\to\infty}} a_n = \inf_{n\geq 1} \sup E_n = \inf_{n\geq 1} \sup_{k\geq n}\{a_k\}.$$

Similarly, the *limit inferior* ($\liminf_{n\to\infty}$ or $\underline{\lim}_{n\to\infty}$) of (a_n) is defined by

$$\underline{\lim}_{n\to\infty} a_n = \sup_{n\geq 1} \inf E_n = \sup_{n\geq 1} \inf_{k\geq n}\{a_k\}.$$

The sequence (a_n) has the limit (or converges to a limit) a iff $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = a$.

The latter definition coincides with the usual definition of the limit of a sequence:

$$\forall \epsilon > 0 \;\; \exists N = N(\epsilon) \;\; s.t. \;\; \forall n \geq N \;\; |a_n - a| < \epsilon.$$

If we let $\alpha_n = \sup\{a_n, a_{n+1}, \dots\}$ and $\beta_n = \inf\{a_n, a_{n+1}, \dots\}$ then $\alpha_n \downarrow$ and $\beta_n \uparrow$ and we have that

$$-\infty \leq \underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} \beta_n \leq \lim_{n\to\infty} \alpha_n = \overline{\lim_{n\to\infty}} a_n \leq \infty.$$

Functions and their limits:

Definition 1.3 A mapping $f : D \to E$ is called a real-valued function from D *into* E if $E \subset \mathbb{R}$ and for every point $x \in D$ there exists a unique point $f(x) \in E$. The set D is called the domain of f ($domf$). The set $ran(f) = \{y \in E : y = f(x) \text{ for some } x \in D\}$ of all possible values of f is called the range of f. If $ran(f) = E$ we say that f is a function from D *onto* E.

(Thus, a function $f(x) = \sin(x)$ defined for any $x \in \mathbb{R}$ is a function from $\mathbb{R}$ *into* $\mathbb{R}$, and is a function from $\mathbb{R}$ *onto* $[-1, 1]$. Also, note that the inverse mapping f^{-1} is multivalued, since it maps $\{1\} \mapsto \{\pi/2 + 2\pi k : k \in \mathbb{Z}\}$, thus, according to the above definition, f^{-1} is not a real-valued function.)

Definition 1.4 Let $\{f_n(x)\}$ $(n = 1, 2, \dots)$ be an infinite sequence of real-valued functions (defined on the same domain). Then we define

$$\overline{\lim} f_n(x) = \inf_{n\geq 1} \sup\{f_n(x), f_{n+1}(x), \dots\},$$

$$\underline{\lim} f_n(x) = \sup_{n\geq 1} \inf\{f_n(x), f_{n+1}(x), \dots\}.$$

If, for some point x from the domain, $\underline{\lim} f_n(x) = \overline{\lim} f_n(x)$, then we say that there exists $\lim_{n\to\infty} f_n(x)$ at x. If $\lim_{n\to\infty} f_n(x) = f(x)$ exists for *every* point $x \in [a, b]$, we say that the sequence *converges pointwise* to f over $[a, b]$.

Definition 1.5 A real-valued function f is said to be *continuous* at x_0 if $\forall \epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Also, f is a continuous function on $\mathbb{R}$, if f is continuous for every $x \in \mathbb{R}$.

The function is called *uniformly continuous* if δ in the above definition does not depend on x_0. (Note that e.g. $f(x) = 1/x$ for $x \in (0, 1)$ is continuous but not uniformly continuous.)

Proposition 1.4 *A real-valued function f is continuous iff for every open interval $U \subset \mathbb{R}$, the set $f^{-1}(U) := \{x : f(x) \in U\}$ is open.*

Definition 1.6 A sequence $\{f_n\}$ converges to f uniformly over the set B if for each $\epsilon > 0$ there exists $N = N(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $x \in B$ and all $n \geq N$. (N here does not depend on the choice of x.)

Note that $f_n \to f$ uniformly on $[a, b]$ iff

$$\sup_{a \leq x \leq b} |f_n(x) - f(x)| \to 0.$$

2 Topology and Metric Spaces

Let E be an arbitrary finite set consisting of n elements, i.e. $E = \{x_1, \ldots, x_n\}$. Then the number of all possible subsets of E is equal to

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Thus, it is often natural to denote the class of all possible subsets of E by 2^E, i.e.

$$2^E := \{\emptyset, \{x_1\}, \ldots, \{x_1, x_2\}, \ldots, \{x_1, \ldots, x_n\}\}.$$

The same notation is often carried over to infinite sets: For any set X, 2^X denotes the class of all possible subsets of X (2^X is called the *power set*).

Definition 2.1 Given a set X, a *topology* on X is a collection $\mathcal{T}$ of subsets of X (i.e. $\mathcal{T} \subset 2^X$) such that
(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(2) $\mathcal{T}$ is closed under finite intersections,
(3) $\mathcal{T}$ is closed under arbitrary unions.

When a topology has been chosen, its members are called *open* sets. Their complements, $F = X \setminus U$, $U \in \mathcal{T}$, are called *closed* sets. The pair $(X, \mathcal{T})$ is called a *topological space.*

Note that for any topological space $(X, \mathcal{T})$, the sets $\emptyset$ and X are at the same time both open and closed. Also, for any set X, the power set 2^X is a topology, called the *discrete topology* on X. For discrete topology, all sets are open and all are closed.

In $\mathbb{R}$, there is a "usual topology", which is generated by the open intervals (a, b). In the "usual" topology, one can show that an arbitrary open set in $\mathbb{R}$ can be represented as a countable union of disjoint open intervals.

Note also that often there exist sets which are neither open nor closed. (Elementary example: $(a, b] \subset \mathbb{R}$ is neither open nor closed in the "usual" topology.)

Definition 2.2 (X, d) is called a *metric space* if X is equipped with a *metric* (or *distance) d*. For any two points $x, y \in X$, $d(x, y)$ is called the *distance* between x and y if d satisfies the following conditions:
(1) $d(x, y) \geq 0$ for all $x, y \in X$;
(2) $d(x, y) = 0$ if and only if $x = y$;
(3) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d is a function satisfying $d(x, x) = 0 \quad \forall x \in X$ and conditions $(1), (3), (4)$, but we permit $d(x, y) = 0$ for some $x \neq y$, then d is called a *pseudometric*.

Examples of Metric spaces:
(1) $(\mathbb{R}, d)$, where

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} .$$

(2) $(\mathbb{R}, d)$, where d is the usual Euclidean distance:

$$d(x, y) = |x - y|.$$

(3) $(C[a, b], d)$, where $C[a, b]$ is the space of continuous real-valued functions on $[a, b]$ and for any two points $x = (x(t))_{t \in [a,b]}$ and $y = (y(t))_{t \in [a,b]}$ in $C[a, b]$,

$$d(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|.$$

(4) $(\mathbb{R}^\infty, d)$, where $\mathbb{R}^\infty$ is a space of infinite sequences and $\forall x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots) \in \mathbb{R}^\infty$,

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Remark: To see a simple example of a *pseudometric*, take a real plane $\mathbb{R}^2 = \{x = (x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$ and introduce a function d on $\mathbb{R}^2$ by defining for any two points $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$,

$$d(x, y) = |x_1 - y_1|.$$

Then, d is a *pseudometric* on $\mathbb{R}^2$. It's not a metric, since it does not distinguish between points (x_1, x_2) and (x_1, x_2^*) where $x_2 \neq x_2^*$.

Definition 2.3 Define an open ball of radius $\epsilon > 0$ centered at $a \in X$ in a metric space (X, d) via:

$$B_\epsilon(a) = \{x \in X : d(x, a) < \epsilon\}.$$

$B_\epsilon(a)$ is also often called an *ϵ-neighborhood* of a.

The natural topology on a metric space (X, d) is the one generated by open balls $B_\epsilon(a)$, $a \in X$, $\epsilon > 0$. (It's called a *metric topology*.) Namely, we'll say that a set $U \subset X$ is open if for every point $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.

Definition 2.4 *For any set $A \in X$, define the set $\overline{A}$, called the closure* of A, by

$$\overline{A} = \{x \in X : \text{ every } \epsilon\text{−neighborhood of } x \text{ contains a point of } A\}.$$

Definition 2.5 A subset A of the metric space (X, d) is called *dense*, if $\overline{A} = X$, i.e. for every point $x \in X$ and $\forall \epsilon > 0$ there exists a point $y \in A$ such that $d(x, y) < \epsilon$.

Examples of dense sets:
(1) The set $\mathbb{Q}$ of rationals is dense in $(\mathbb{R}, d)$, where d is the usual Euclidean distance.
(2) Let Π denote the family of *all* polynomials with real coefficients, i.e.

$$\Pi = \{p_k(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k : k \in \mathbb{Z}^+, a_0, \ldots, a_k \in \mathbb{R}\}.$$

Then the Weierstrass Approximation theorem states: Every $f \in C[a, b]$ can be uniformly approximated by a sequence of polynomials, i.e. for every $\epsilon > 0$ there exists $p \in \Pi$ such that

$$d(f, p) := \sup_{x \in [a,b]} |f(x) - p(x)| < \epsilon.$$

In other words, Π is dense in $(C[a, b], d)$ (where d is the usual supremum metric).

Definition 2.6 A metric space (X, d) is called *separable* if there exists a *countable dense* subset.

Definition 2.7 A sequence $(x_n)_{n\geq 1}$ in a metric space (X, d) is called *Cauchy* if $\forall \epsilon > 0\ \exists N = N(\epsilon)\ \forall m, n \geq N\ d(x_m, x_n) < \epsilon$. Or, equivalently, one may require that $d(x_m, x_n) \to 0$ as $m, n \to \infty$.

In $\mathbb{R}$ (with the Euclidean metric), a sequence $(x_n)_{n\geq 1}$ is convergent if and only if it is a Cauchy sequence. In a general metric space, every convergent sequence is certainly Cauchy, but the converse statement is often false. In other words, there may exist a Cauchy sequence $(x_n)_{n\geq 1}$ in (X, d), for which there are **no** points $y \in X$ with the property $d(x_n, y) \to 0$ as $n \to \infty$.

Definition 2.8 A metric space (X, d) is called *complete* if every Cauchy sequence $(x_n)_{n\geq 1} \in X$ is convergent in it, i.e. $\exists x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. If the latter fails, (X, d) is called *incomplete.*

Examples:
(1) A metric space (X, d), with d given by

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases}$$

is always complete (regardless of the choice of X).
(2) The space $(C[0, 1], d)$, where $d(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)|$, is complete.

(3) The space $(\mathbb{Q}, d)$, where d is a Euclidean distance, is incomplete. (Eg., take $x_n = \sum_{i=0}^{n} \frac{1}{i!} \in \mathbb{Q}$, then $(x_n)_{n\geq 1}$ is a Cauchy sequence in $(\mathbb{Q}, d)$, but the limit point e $\notin \mathbb{Q}$.)