Stat 620, Fall 2005
September 13

1 Real line and its subsets

2 Topology and metric spaces

3 Classes of sets (*continued*)

Remark: Different classes $\mathcal{C}$ may generate the same σ-field. To see this, take $\mathcal{C}_1 \subset \sigma(\mathcal{C}_2)$ and $\mathcal{C}_2 \subset \sigma(\mathcal{C}_1)$, then $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2)$. In particular, it's easy to show that $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{C}^*)$, where we define $\mathcal{C}^* = \{(a, b] : a, b \in \mathbb{R}\}$. In fact, it follows immediately from the equation: $(-\infty, c] = \bigcup_{n=1}^{\infty}(-n, c]$.

Definition 3.1 Let $\{E_n\}_{n=1}^{\infty}$ be an arbitrary sequence of subsets of Ω. Then define

$$\overline{\lim} E_n := \{x \in \Omega : x \in E_n \ \text{ for infinitely many } \ n \in \mathbb{N}\}$$

and

$$\underline{\lim} E_n := \{x \in \Omega : x \in E_n \ \text{ for all but finitely many } \ n \in \mathbb{N}\}.$$

Note: $\underline{\lim} E_n \subset \overline{\lim} E_n$ and
(i) $x \in \overline{\lim} E_n$ iff $\forall n \in \mathbb{N}\ \exists m \geq n$ s.t. $x \in E_m$;
(ii) $x \in \underline{\lim} E_n$ iff $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N\ x \in E_n$.

Theorem 3.1 *For any sequence* $\{E_n\} \subset \Omega$,

$$(i)\ \ \overline{\lim}\, E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m;$$

$$(ii)\ \ \underline{\lim}\, E_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m.$$

Proof: Let us show (ii). Suppose $x \in \underline{\lim} E_n$, then $\exists N$ s.t. for all $m \geq N$, $x \in E_m$. Therefore, $x \in \bigcap_{m=N}^{\infty}$ for some $N \in \mathbb{N}$, which implies that $x \in \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m$.
Conversely, suppose $x \in \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m$, then $\exists N \in \mathbb{N}$ s.t. $x \in \bigcap_{m=N}^{\infty} E_m$. Therefore, for all $m \geq N$, $x \in E_m$, thus $x \in \underline{\lim}\, E_n$. Proof of (i) is left as an exercise.

Definition 3.2 A sequence $\{E_n\}$ is called *convergent* if $\underline{\lim}\, E_n = \overline{\lim}\, E_n$.

Definition 3.3 *A sequence* $\{E_n\}_{n=1}^{\infty}$ *is called a monotone increasing (decreasing) sequence if* $E_n \subset E_{n+1}$ *(*$E_n \supset E_{n+1}$*, respectively) for all* n*.*

Theorem 3.2 *A monotone increasing (decreasing) sequence is convergent and*

$$\lim_n E_n = \bigcup_{n=1}^{\infty} E_n \qquad \left(\lim_n E_n = \bigcap_{n=1}^{\infty} E_n, \ \textit{respectively} \right).$$

Proof: Suppose $\{E_n\}$ is monotone increasing, then for any given $N \geq 1$ and $\forall k \leq N$, $\bigcup_{m=k}^{N} E_m = E_N$. Thus, $\bigcup_{m=k}^{\infty} E_m$ does not depend on k. Therefore,

$$\overline{\lim} E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_m = \bigcup_{m=1}^{\infty} E_m.$$

On the other hand,

$$\underline{\lim} E_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m = \bigcup_{n=1}^{\infty} E_n.$$

Thus, $\underline{\lim} E_n = \overline{\lim} E_n = \bigcup_{n=1}^{\infty} E_n$. The proof for a monotone decreasing sequence is analogous.

Definition 3.4 For any set E, the *indicator function* of E is given by

$$\chi_E(x) \equiv I_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E. \end{cases}$$

Properties:
(a) $\chi_E(x) \leq \chi_F(x)\ \forall x \iff E \subset F$.
(b) $\chi_E(x) = \chi_F(x)\ \forall x \iff E = F$.
(c) $\chi_\emptyset \equiv 0$; $\chi_\Omega \equiv 1$.
(d) $\chi_{\bigcap_{i=1}^n E_i}(x) = \Pi_{i=1}^n \chi_{E_i}(x) = \min_{1\leq i\leq n} \chi_{E_i}(x)$; $\quad \chi_{\bigcup_{i=1}^n E_i}(x) = \max_{1\leq i\leq n} \chi_{E_i}(x)$.
(e) for disjoint $\{E_i\}_{i=1}^n$, $\quad \chi_{\bigcup_{i=1}^n E_i}(x) = \sum_{i=1}^{n} \chi_{E_i}(x)$.

Definition 3.5 A *ring* is a non-empty class $\mathcal{R}$ of subsets of the space Ω, s.t. $E, F \in \mathcal{R} \Longrightarrow E \cup F \in \mathcal{R}$ and $E - F \in \mathcal{R}$.

Theorem 3.3 *A ring contains* $\emptyset$*. A ring is closed under the formations of finite unions and finite intersections.*

Proof: Take arbitrary $E \in \mathcal{R}$, then $\emptyset = E - E \in \mathcal{R}$. Note also that $\forall F \in \mathcal{R}$, $E \cap F = E - (E - F) \in \mathcal{R}$, then proceed by induction.

Alternative definitions: Let $\mathcal{R}$ be a nonempty class of subsets of Ω which is closed under formation of
(i) unions and proper differences; or
(ii) intersections, proper differences and disjoint unions.
(A difference $A - B$ is called *proper* if $B \subset A$.)
Then $\mathcal{R}$ is a ring.
Proof: (i) follows from $E - F = (E \cup F) - F$.
(ii) follows from $E \cup F = (E - (E \cap F)) \cup (F - (E \cap F)) \cup (E \cap F)$.

Theorem 3.4 *A field is a ring of which the entire space Ω is a member, and conversely.*

Definition 3.6 A *semiring* is a class $\mathcal{P}$ of subsets of Ω such that
(i) $\emptyset \in \mathcal{P}$;
(ii) $\forall E, F \in \mathcal{P} \Longrightarrow E \cap F \in \mathcal{P}$;
(iii) $\forall E, F \in \mathcal{P}$, then $E - F = \displaystyle\bigcup_{i=1}^{n} E_i$ for some $n \in \mathbb{N}$, where $\{E_i\}_{i=1}^{n}$ are disjoint sets of $\mathcal{P}$.

Example: Let $\mathcal{P} = \{(a, b] : a, b \in \mathbb{R}\}$. Then $\mathcal{P}$ is a semiring (but not a ring).

Proposition 3.1 *For any class $\mathcal{C}$ there exists a unique smallest ring $\mathcal{R}(\mathcal{C})$ containing $\mathcal{C}$.*

(We'll say that $\mathcal{R}(\mathcal{C})$ is the ring generated by $\mathcal{C}$.)
Idea behind the proof: Take the intersection of all the rings containing $\mathcal{C}$ and show that such an intersection is a ring.

Question: Does there exist a (unique) smallest semiring containing a given class $\mathcal{C}$? No.

Proposition 3.2 *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets of a ring $\mathcal{R}$, and $E = \bigcup_{n=1}^{\infty} E_n$ (E may not be in $\mathcal{R}$). Then*
(i) $E = \bigcup_{n=1}^{\infty} F_n = \lim F_n$, where $F_n = \bigcup_{m=1}^{n} E_m$ $(n = 1, 2, \ldots)$ form a monotone increasing sequence of subsets of $\mathcal{R}$;
(ii) $E = \bigcup_{n=1}^{\infty} G_n$, where G_n's are disjoint sets of $\mathcal{R}$ such that $G_n \subset E_n$ $\forall n$.

Proof: (i) is obvious. To prove (ii), let $G_1 = E_1$ and $G_n = E_n - \bigcup_{j=1}^{n-1} E_j$ for all $n \geq 2$. Clearly $G_n \subset E_n$ for all n. Since G_n is disjoint from $E_1, \ldots, E_{n-1}$, we have that G_n is disjoint from $G_1, \ldots, G_{n-1}$. Moreover, $G_n \in \mathcal{R}$ and $\bigcup_{n=1}^{\infty} G_n = E$. Thus, (ii) holds.

Definition 3.7 A ring $\mathcal{R}$ is called a *sigma-ring* if $\mathcal{R}$ is closed under countable unions.

It's easy to show the existence of a σ-ring generated by a given class $\mathcal{C}$. Let us denote it by $\mathcal{S}(\mathcal{C})$.

Proposition 3.3 *(i) Let $\mathcal{C}, \mathcal{C}_0$ be two classes of subsets of Ω with $\mathcal{C}_0 \subset \mathcal{C}$, then $\mathcal{S}(\mathcal{C}_0) \subset \mathcal{S}(\mathcal{C})$ and $\sigma(\mathcal{C}_0) \subset \sigma(\mathcal{C})$; (ii) $\forall \mathcal{E} \subset 2^\Omega$, $\mathcal{S}(\mathcal{R}(\mathcal{E})) = \mathcal{S}(\mathcal{E})$.*

Proof: To prove (i), note that $\mathcal{C}_0 \subset \mathcal{C} \subset \mathcal{S}(\mathcal{C}) \subset \sigma(\mathcal{C})$, thus $\mathcal{S}(\mathcal{C}_0) \subset \mathcal{S}(\mathcal{C})$ and $\sigma(\mathcal{C}_0) \subset \sigma(\mathcal{C})$. To prove (ii), note that $\mathcal{R}(\mathcal{E}) \supset \mathcal{E}$, implying that $\mathcal{S}(\mathcal{R}(\mathcal{E}))$ is a σ-ring containing $\mathcal{E}$, thus $\mathcal{S}(\mathcal{R}(\mathcal{E})) \supset \mathcal{S}(\mathcal{E})$. On the other hand, $\mathcal{S}(\mathcal{E})$ is a ring containing $\mathcal{E}$, thus $\mathcal{S}(\mathcal{E}) \supset \mathcal{R}(\mathcal{E})$. Therefore, $\mathcal{S}(\mathcal{E})$ is a σ-ring containing $\mathcal{R}(\mathcal{E})$, and we obtain that $\mathcal{S}(\mathcal{E}) \supset \mathcal{S}(\mathcal{R}(\mathcal{E}))$. Thus, $\mathcal{S}(\mathcal{E}) = \mathcal{S}(\mathcal{R}(\mathcal{E}))$.

Proposition 3.4 *If $\mathcal{C}$ is a non-empty class of subsets of Ω, any set in $\mathcal{R}(\mathcal{C})$ can be covered by a finite union of sets in $\mathcal{C}$. That is, if $E \in \mathcal{R}(\mathcal{C})$ then $\exists n \in \mathbb{N}$, $\exists F_1, \ldots, F_n \in \mathcal{C}$ such that $E \subset \bigcup_{i=1}^n F_i$.*

Proof: Let $\mathcal{G}$ be the class of those sets which can be covered by some finite union of sets from $\mathcal{C}$. It's easy to show that $\mathcal{G}$ is a ring containing $\mathcal{C}$. Thus, $\mathcal{G} \supset \mathcal{R}(\mathcal{C})$.

Proposition 3.5 *Let $\mathcal{P}$ be a semiring. Then $\mathcal{R}(\mathcal{P}) = \mathcal{L}$, where*

$$\mathcal{L} := \left\{ E = \bigcup_{i=1}^n E_i : \{E_i\}_{i=1}^n \in \mathcal{P} \text{ are disjoint}, n \in \mathbb{N} \right\}.$$

Proof: $E \in \mathcal{L} \Longrightarrow E \in \mathcal{R}(\mathcal{P})$. Hence, $\mathcal{L} \subset \mathcal{R}(\mathcal{P})$. To show the opposite inclusion, it is sufficient to show that $\mathcal{L}$ is a ring. Note that $\mathcal{L}$ is closed under the formation of (finite) disjoint unions. Moreover, $\mathcal{L}$ is closed under the formation of finite intersections, since $\forall E, F \in \mathcal{L}$, there exist disjoint sets $\{E_i\}_{i=1}^n \in \mathcal{P}$, and disjoint sets $\{F_j\}_{j=1}^m \in \mathcal{P}$, s.t. $E = \bigcup_{i=1}^n E_i$ and $F = \bigcup_{j=1}^m F_j$, and

$$E \cap F = \bigcup_{i=1}^n \bigcup_{j=1}^m (E_i \cap F_j),$$

where $(E_i \cap F_j) \in \mathcal{P}$ and the sets $(E_i \cap F_j)$ are disjoint for $(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$. Finally, $\mathcal{L}$ is closed under the formation of differences, since

$$E - F = \cup_{i=1}^n \left(E_i - \cup_{j=1}^m F_j \right) = \cup_{i=1}^n \cap_{j=1}^m (E_i - F_j),$$

where $\cap_{j=1}^m (E_i - F_j)$ are disjoint sets $(i = 1, \ldots, n)$. Thus, if we can show that $E_i - F_j \in \mathcal{L}$, then (by the above proved properties of $\mathcal{L}$) we will have that $E - F \in \mathcal{L}$. But $E_i, F_j \in \mathcal{P}$, thus, by definition of a semiring, there exist $\{A_l\}_{l=1}^L \in \mathcal{P}$ s.t. $E_i - F_j = \bigcup_{l=1}^L A_l$ and $\{A_l\}_{l=1}^L$ are disjoint. Thus, $\forall i, j$, we have that $E_i - F_j \in \mathcal{L}$ and the desired result follows.

■

The following classes of sets are widely used and may be useful to us later on.

Definition 3.8 $\mathcal{M}$ is a *monotone* class of subsets of Ω if for every monotone increasing or decreasing sequence $\{E_n\}_{n=1}^{\infty} \in \mathcal{M}$, $\lim E_n \in \mathcal{M}$.

Remark: Any σ-field is a monotone class. Conversely, if a monotone class is a field, then it's a σ-field. Also, take an arbitrary $\mathcal{C} \subset 2^{\Omega}$. Then 2^{Ω} is a monotone class containing $\mathcal{C}$. The intersection of all monotone classes containing $\mathcal{C}$ is again a monotone class and it's the smallest monotone class containing $\mathcal{C}$ (i.e. it's the monotone class *generated* by $\mathcal{C}$, denoted by $\mathcal{M}(\mathcal{C})$).

Proposition 3.6 *If $\mathcal{F}$ is a field then $\mathcal{M}(\mathcal{F}) = \sigma(\mathcal{F})$.*

Definition 3.9 A class $\mathcal{D}$ of subsets of Ω is called a *D-system (Dynkin system, or λ-system)* if the following conditions hold:
(i) $\Omega \in \mathcal{D}$;
(ii) $\forall E, F \in \mathcal{D}$ s.t. $E \subset F$ implies that $F - E \in \mathcal{D}$;
(iii) $\forall \{E_n\} \in \mathcal{D}$ s.t. $E_n \uparrow E$ implies that $E \in \mathcal{D}$.

Exercise: Show that a class $\mathcal{D}$ is a D-system iff $\mathcal{D}$ contains Ω and is closed under the formation of complements and countable *disjoint* unions.

Definition 3.10 A non-empty class $\mathcal{E}$ of subsets of Ω is called a π-*system*, if $\mathcal{E}$ is closed under the formation of finite intersections.