Stat 620, Fall 2005

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1 Real line and its subsets

2 Topology and metric spaces

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4 Measures

4.1 Measure on a semiring. Extension to a ring

Let $\Omega \neq \emptyset$ and let $\mathcal{C}$ be a class of subsets of Ω. A function $\mu : \mathcal{C} \to E$ is called an E-valued *set* function. Let us focus on the case when $E = \overline{\mathbb{R}} = [-\infty, +\infty]$.

Definition 4.1 A set function μ defined on $\mathcal{C}$ is called *additive* if for all $E, F \in \mathcal{C}$ s.t. $E \cup F \in \mathcal{C}$ and $E \cap F = \emptyset$, $\mu(E \cup F) = \mu(E) + \mu(F)$.

Definition 4.2 μ defined on $\mathcal{C}$ is called *finitely* (*countably*) *additive* if $\mu(\bigcup\limits_{i=1}^{n} E_i) = \sum\limits_{i=1}^{n} \mu(E_i)$ $\left(\mu(\bigcup\limits_{i=1}^{\infty} E_i) = \sum\limits_{i=1}^{\infty} \mu(E_i)\right)$ for all disjoint sets $E_i \in \mathcal{C}$, $i = 1, 2, \ldots$, whose union $\bigcup\limits_{i=1}^{n} E_i \in \mathcal{C}$ $\left(\bigcup\limits_{i=1}^{\infty} E_i \in \mathcal{C}\right)$.

Definition 4.3 μ is called a *finite* set function on $\mathcal{C}$ if $|\mu(E)| < \infty$ $\forall E \in \mathcal{C}$. μ is called σ-finite on $\mathcal{C}$ if $\forall E \in \mathcal{C}$ $\exists \{E_n\}_{n=1}^{\infty} \in \mathcal{C}$ s.t. $E \subset \bigcup_{n=1}^{\infty} E_n$ and $|\mu(E_n)| < \infty$.

Definition 4.4 Let μ, ν be two set functions defined on classes $\mathcal{C}_0$, $\mathcal{C}$ respectively. Suppose $\mathcal{C}_0 \subset \mathcal{C}$ and $\mu(E) = \nu(E)$ $\forall E \in \mathcal{C}_0$, then ν is said to be an *extension* of μ to $\mathcal{C}$, and μ is said to be a *restriction* of ν to $\mathcal{C}_0$ (written $\nu|_{\mathcal{C}_0} = \mu$).

Definition 4.5 A *measure* is a non-negative, countably additive set function μ defined on a semi-ring $\mathcal{P}$ s.t. $\mu(\emptyset) = 0$.

Note: $\mu(\emptyset) = 0$ in the above definition follows from countable additivity except in the case when $\mu(E) = \infty$ $\forall E \in \mathcal{P}$, since if $\exists E \in \mathcal{P}$ s.t. $\mu(E) < \infty$, then $E = E \cup \emptyset \cup \cdots \cup \emptyset$, thus $\mu(E) = \mu(E) + \mu(\emptyset) + \mu(\emptyset) + \ldots$, implying that $\mu(\emptyset) = 0$.

If a measure μ, as a set function on $\mathcal{P}$, is finite (or σ-finite), μ is referred to as a finite (σ-finite), measure.

It is often convenient to define a set function, with certain properties, on a small class $\mathcal{C}$ of sets, and then extend it to obtain a set function on a much larger class of sets (like σ-ring $\mathcal{S}(\mathcal{C})$).

Example: Consider a semiring $\mathcal{P} = \{[a,b) : a,b \in \mathbb{R}\}$ and define a set function $\mu : \mathbb{R} \to \overline{\mathbb{R}}$ by

$$\mu([a,b)) = |b-a| = \text{ length of } [a,b).$$

Then μ is a (finite) measure on $\mathcal{P}$ and one wants to find a natural extension of μ to a ring $\mathcal{R}(\mathcal{P})$. By a proposition proved earlier,

$$\mathcal{R}(\mathcal{P}) = \{\bigcup_{i=1}^{n}[a_i,b_i) : [a_i,b_i) \cap [a_j,b_j) = \emptyset \;\; \forall i \neq j, \; \forall n \in \mathbb{N}\},$$

and it's natural to extend μ to $\mathcal{R}(\mathcal{P})$ by defining

$$\nu\left(\bigcup_{i=1}^{n}[a_i,b_i)\right) = \sum_{i=1}^{n}\mu([a_i,b_i))$$

for all $n \in \mathbb{N}$ and all disjoint intervals $[a_1,b_1), \ldots, [a_n,b_n) \subset \mathbb{R}$.
Similarly, the extension of μ to $\mathcal{R}(\mathcal{P})$ is constructed for the case when $\mathcal{P}$ is an arbitrary semiring.

Theorem 4.1 *If μ is a measure on a semiring $\mathcal{P}$, then there exists a unique extension ν of μ to $\mathcal{R}(\mathcal{P})$. Moreover, ν is a measure on $\mathcal{R}(\mathcal{P})$ and if μ is finite (or σ-finite) on $\mathcal{P}$, then ν is finite (σ-finite) on $\mathcal{R}(\mathcal{P})$.*

Remark on Sums: Suppose Ω is a countable set, say $\Omega = \{\omega_1, \omega_2, \ldots\}$, and let $f : \Omega \to [0,\infty)$ be such that the limit

$$s := \lim_{n\to\infty}(f(\omega_1) + f(\omega_2) + \cdots + f(\omega_n)) < \infty.$$

Suppose also that $\Omega = \{\omega_1^{'}, \omega_2^{'}, \ldots\}$, then

$$\lim_{n\to\infty}\left(f(\omega_1^{'}) + f(\omega_2^{'}) + \cdots + f(\omega_n^{'})\right) = s.$$

(I.e. the order of summation of non-negative terms does not matter.)

Since the order of summation of non-negative terms does not matter, we will often write simply

$$\sum_{\omega\in\Omega} f(\omega)$$

for a countable Ω and a non-negative f.

Proposition 4.1 *For an arbitrary measure μ on a ring $\mathcal{R}$, the following properties hold:*
(i) μ is monotone, i.e. $\forall E, F \in \mathcal{R}$ such that $E \subset F$, $\mu(E) \leq \mu(F)$.
(ii) μ is subtractive, i.e. $\forall E, F \in \mathcal{R}$ such that $E \subset F$, $\mu(F - E) = \mu(F) - \mu(E)$.
(iii) μ is countably subadditive on $\mathcal{R}$, i.e. $\forall \{E_i\}_{i=1}^{\infty} \in \mathcal{R}$ such that $(\bigcup_{i=1}^{\infty} E_i) \in \mathcal{R}$, the following inequality holds:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

4.2 Outer measures

Definition 4.6 A monotone set function $\mu^* : 2^{\Omega} \to [0, +\infty]$ is called an *outer measure* if the following conditions hold:
(i) $\mu^*(\emptyset) = 0$, and
(ii) μ^* is countably subadditive on 2^{Ω}, i.e. $\forall \{E_i\}_{i=1}^{\infty} \in 2^{\Omega}$, $\mu^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Definition 4.7 Given an outer measure μ^*, define a class

$$S_{\mu^*} := \{E \in 2^{\Omega} : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \ \ \forall A \in 2^{\Omega}\}.$$

The sets belonging to S_{μ^*} are called *μ^*-measurable.*

Note that $\emptyset$ and Ω are μ^*-measurable for any outer measure μ^* on 2^{Ω}.

Simple examples of outer measures:
(1) Put $\mu^*(E) \equiv 0$ for all $E \in 2^{\Omega}$. Then μ^* is an outer measure and $S_{\mu^*} = 2^{\Omega}$.
(2) Take $\Omega = \{a, b\}$ and put $\mu^*(\emptyset) = 0$, $\mu^*(\{a\}) = \mu^*(\{b\}) = 1$, $\mu^*(\Omega) = 3/2$. Then μ^* is an outer measure and $S_{\mu^*} = \{\emptyset, \Omega\}$.
(3) Take $\Omega = \{a, b, c\}$ and put $\mu^*(\emptyset) = 0$, $\mu^*(\{a\}) = 1$, $\mu^*(\{b, c\}) = 2$, $\mu^*(\{b\}) = \mu^*(\{c\}) = 3/2$, $\mu^*(\{a, b\}) = \mu^*(\{a, c\}) = 5/2$, $\mu^*(\Omega) = 3$. Then μ^* is an outer measure and $S_{\mu^*} = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$.

Proposition 4.2 *$\forall E, F \in S_{\mu^*}$, $\forall A \subset \Omega$,*
(i) $\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c)$;
(ii) $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E \cap F^c)$;
(iii) If $E \cap F = \emptyset$, then $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E) + \mu^*(A \cap F)$.

Note: direct comparison of *(ii)* and *(iii)* in the above proposition leads to the following equation: $\forall E, F \in S_{\mu^*}$ and $\forall A \subset \Omega$,

$$\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c). \tag{1}$$

Theorem 4.2 *S_{μ^*} is a σ-field. If $\{E_n\}_{n=1}^\infty \in S_{\mu^*}$ is a disjoint sequence and $E = \bigcup_{i=1}^\infty E_i$, then $\mu^*(E) = \sum_{i=1}^\infty \mu^*(E_i)$. Thus $\mu^*|_{S_{\mu^*}}$ is a measure on S_{μ^*}.*

Given a measure μ on a ring $\mathcal{R}$, our objective now is to find an outer measure μ^* with the property that $\mu^*|_{\mathcal{R}} = \mu$.

Definition 4.8 Given an outer measure μ on a ring $\mathcal{R}$, define a set function ν_μ such that $\forall E \subset \Omega$,

$$\nu_\mu(E) := \begin{cases} \inf\left\{\sum_{i=1}^\infty \mu(E_i) : (\bigcup_{i=1}^\infty E_i) \supset E, E_i \in \mathcal{R}\ \forall i = 1, 2, \dots\right\}, & \text{if } \exists \{E_i\}_{i=1}^\infty \in \mathcal{R} \text{ s.t. } E \subset \bigcup_{i=1}^\infty E_i, \\ +\infty, & \text{otherwise.} \end{cases}$$

Theorem 4.3 *ν_μ is an outer measure and $\nu_\mu|_{\mathcal{R}} = \mu$.*

4.3 Extension of measures from rings to generated σ-rings

Let μ be a measure on a ring $\mathcal{R}$ and ν_μ be an outer measure generated by μ, given by Definition 4.8. Let S_{ν_μ} be the corresponding σ-field of ν_μ-measurable sets.

Proposition 4.3 *$\mathcal{R} \subset S_{\nu_\mu}$, implying that $\mathcal{S}(\mathcal{R}) \subset S_{\nu_\mu}$ and $\sigma(\mathcal{R}) \subset S_{\nu_\mu}$.*

Thus, we can immediately deduce that $\nu_\mu|_{\mathcal{S}(\mathcal{R})}$ is a measure extending μ on $\mathcal{R}$ to $\mathcal{S}(\mathcal{R})$.
Is the extension to $\mathcal{S}(\mathcal{R})$ unique? The answer is 'yes' if μ is σ-finite on $\mathcal{R}$. The proof of uniqueness requires the following fact:

Proposition 4.4 *Let $\mathcal{C}$ be a non-empty class of subsets of Ω and let $\mathcal{D}^\circ(\mathcal{C})$ be the smallest class containing $\mathcal{C}$ which is closed under the formation of proper differences and countable disjoint unions. If $\mathcal{C}$ is closed under the formation of (finite) intersections, then $\mathcal{D}^\circ(\mathcal{C}) = \mathcal{S}(\mathcal{C})$.*

Theorem 4.4 *Let μ be a measure on a ring $\mathcal{R}$. Then there exists a measure $\tilde{\mu}$ on $\mathcal{S}(\mathcal{R})$ extending μ. If μ is σ-finite on $\mathcal{R}$, then $\tilde{\mu}$ is the unique such extension of μ to $\mathcal{S}(\mathcal{R})$ and is itself σ-finite on $\mathcal{S}(\mathcal{R})$.*

Remark: We have seen that $\nu_\mu|_{\sigma(\mathcal{R})}$ is a measure on $\sigma(\mathcal{R})$ extending measure μ from the ring $\mathcal{R}$. Suppose that μ is σ-finite. Is the extension of μ to $\sigma(\mathcal{R})$ unique? In general, the answer is 'no'. For example, take $\Omega = \{a, b, c\}$ and $\mathcal{R} = \{\emptyset, \{a, b\}\}$. Define a measure μ on $\mathcal{R}$ by $\mu(\emptyset) = 0$, $\mu(\{a, b\} = 1$. Consider $\mathcal{F}(\mathcal{R}) = \sigma(\mathcal{R}) = \{\emptyset, \Omega, \{a, b\}, \{c\}\}$ and put $\nu_1(\emptyset) = 0$, $\nu_1(\{a, b\}) = 1$, $\nu_1(\{c\}) = 0$ and $\nu_1(\Omega) = 1$. Also, put $\nu_2(\emptyset) = 0$, $\nu_2(\{a, b\}) = 1$, $\nu_2(\{c\}) = 1$ and $\nu_2(\Omega) = 2$. Then clearly, $\nu_1 = \nu_2 = \mu$ on $\mathcal{R}$ (and $\mathcal{S}(\mathcal{R})$), but $\nu_1 \neq \nu_2$ on $\mathcal{F}(\mathcal{R}) = \sigma(\mathcal{R})$.

Summary: We showed that starting with a measure μ on a semiring $\mathcal{P}$, an extension may be obtained to a measure ν on $\mathcal{R}(\mathcal{P})$. ν can then be extended to a measure $\tilde{\mu}$ on $\mathcal{S}(\mathcal{P}) = \mathcal{S}(\mathcal{R}(\mathcal{P}))$. The extension from μ to ν is unique. The extension from ν to $\tilde{\mu}$ is unique, provided ν is σ-finite on $\mathcal{R}$. This will be so if μ is σ-finite on $\mathcal{P}$. Thus,

Theorem 4.5 *Let μ be a measure on a semiring $\mathcal{P}$. Then there exists a measure $\tilde{\mu}$ on $\mathcal{S}(\mathcal{P})$, extending μ on $\mathcal{P}$ ($\tilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{P}$). If μ is σ-finite on $\mathcal{P}$, then $\tilde{\mu}$ is the unique extension to $\mathcal{S}(\mathcal{P})$ and is itself σ-finite on $\mathcal{S}(\mathcal{P})$.*

Lebesgue Measure: Take $\Omega = \mathbb{R}$ and let $\mathcal{P} = \{(a, b] : -\infty < a \leq b < \infty\}$, and define $\forall (a, b] \in \mathcal{P}$,

$$\lambda \{(a, b]\} = b - a.$$

Suppose we can show that λ is a σ-finite measure on $\mathcal{P}$. Then, by previous theorem, there exists a unique extension $\tilde{\lambda}$ of λ to $\mathcal{S}(\mathcal{P})$. Note that $\mathcal{S}(\mathcal{P})$ should contain

$$\mathbb{R} = \bigcup_{n=-\infty}^{+\infty} (n, n+1].$$

Therefore, $\mathcal{S}(\mathcal{P}) = \sigma(\mathcal{P}) = \mathcal{B}_{\mathbb{R}}$. Measure $\tilde{\lambda}$ (uniquely) defined on the class of Borel sets is called a *Lebesgue measure.*

Lebesgue-Stieltjes Measures: Take $\Omega = \mathbb{R}$ and let F be any finite-valued, non-decreasing function on $\mathbb{R}$, such that F is right-continuous at all points. Let $\mathcal{P} = \{(a, b] : -\infty < a \leq b < \infty\}$ and define $\forall (a, b] \in \mathcal{P}$,

$$\lambda_F \{(a, b]\} = F(b) - F(a).$$

Then, by similar arguments, one can show that there exists a unique extension $\tilde{\lambda}_F$ on $\mathcal{B}_{\mathbb{R}}$ of a given measure λ_F on $\mathcal{P}$. Such a $\tilde{\lambda}_F$ is called a *Lebesgue-Stieltjes measure* on $\mathcal{B}_{\mathbb{R}}$ corresponding to the function F.

Probability measures: A measure μ defined on a field $\mathcal{F}$ is called a *probability measure* if $\mu(\Omega) = 1$. Then, by the preceding theorem, there exists a unique extension $\tilde{\mu}$ on $\sigma(\mathcal{F}) = \mathcal{S}(\mathcal{F})$ which is a probability measure on $\sigma(\mathcal{F})$.