Stat 620, Fall 2005
September 22

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In general, given a measure μ on a σ-ring $\mathcal{S}$, it is often possible to have a set $E \in \mathcal{S}$ with $\mu(E) = 0$, but whose subset F is not in $\mathcal{S}$, i.e. $F \subset E$ and $F \notin \mathcal{S}$. Then, μ is not defined for such subsets F of sets of measure zero. One of the objectives of this section is to "enlarge" or "complete" the σ-ring $\mathcal{S}$ to include *all* subsets of sets of measure *zero*, and extend μ to a measure on that completed σ-ring.

Definition 4.4.1 If for all $E \in \mathcal{S}$ with $\mu(E) = 0$, every subset $F \subset E$ satisfies the condition $F \in \mathcal{S}$, then measure μ on the σ-ring $\mathcal{S}$ is called *complete.* Otherwise, μ is called *incomplete* on $\mathcal{S}$.

We will show that a measure μ on a σ-ring $\mathcal{S}$ may be *completed* by slightly enlarging the σ-ring.

Theorem 4.4.1 *Let μ be a measure on a σ-ring $\mathcal{S}$. Define the class $\overline{\mathcal{S}}$ by*

$$\overline{\mathcal{S}} = \{E \cup N : E \in \mathcal{S}, N \subset A \text{ for some } A \in \mathcal{S} \text{ with } \mu(A) = 0\}.$$

Then $\overline{\mathcal{S}}$ is a σ-ring. A measure $\overline{\mu}$ may be defined on $\overline{\mathcal{S}}$ by

$$\overline{\mu}(E \cup N) = \mu(E), \text{ where } E \in \mathcal{S},\ N \subset A,\ A \in \mathcal{S},\ \mu(A) = 0.$$

$\overline{\mu}$ is then a complete measure on $\overline{\mathcal{S}}$, extending μ on $\mathcal{S}$.

Thus, a measure μ on a σ-ring may be extended to the "slightly larger" σ-ring $\overline{\mathcal{S}}$ to give a complete measure, called the *completion* of μ. The completion is unique on $\overline{\mathcal{S}}$. In fact the following result holds:

Proposition 4.4.1 *Let μ be any measure on a σ-ring $\mathcal{S}$, and $\overline{\mu}$ be the completion of μ on the σ-ring $\overline{\mathcal{S}}$. If ν is an extension of μ to a complete measure on a σ-ring $\mathcal{T}$ ($\mathcal{T} \supset \mathcal{S}$), then $\mathcal{T} \supset \overline{\mathcal{S}}$, and ν extends $\overline{\mu}$ (i.e. $\nu(E) = \overline{\mu}(E)$ when $E \in \overline{\mathcal{S}}$).*

Remark: Consider the Lebesgue measure λ defined on $\mathcal{B}_{\mathbb{R}}$ (the class of Borel sets). Then its completion $\overline{\lambda}$ on $\overline{\mathcal{B}}_{\mathbb{R}}$ is also called a Lebesgue measure, whereas the class

$$\overline{\mathcal{B}}_{\mathbb{R}} = \{E \cup N : E \in \mathcal{B}_{\mathbb{R}}, N \subset A \in \mathcal{B}_{\mathbb{R}}, \lambda(A) = 0\}$$

is called the class of *Lebesgue measurable* sets.

Theorem 4.4.2 *If μ is a measure on a ring $\mathcal{R}$, and $\{E_n\}_{n=1}^{\infty}$ is a monotone increasing sequence of sets in $\mathcal{R}$ such that $\lim E_n \in \mathcal{R}$, then $\mu(\lim E_n) = \lim_{n\to\infty} \mu(E_n)$.*

Theorem 4.4.3 *If μ is a measure on a ring $\mathcal{R}$, and $\{E_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of sets in $\mathcal{R}$ of which at least one has finite measure, and if $\lim E_n \in \mathcal{R}$, then $\mu(\lim E_n) = \lim_{n\to\infty} \mu(E_n)$.*

Definition 4.4.2 A set function μ defined on a class $\mathcal{C}$ is said to be *continuous from below at a set $E \in \mathcal{C}$* if for every monotone increasing sequence $\{E_n\}_{n=1}^{\infty} \in \mathcal{C}$ such that $\lim E_n = E$, we have that $\lim_{n\to\infty} \mu(E_n) = \mu(E)$. A set function μ defined on a class $\mathcal{C}$ is said to be *continuous from above at $E \in \mathcal{C}$* if for every monotone decreasing sequence $\{E_n\}_{n=1}^{\infty} \in \mathcal{C}$ such that $\lim E_n = E$ and $|\mu(E_m)| < \infty$ for some $m \in \mathbb{N}$, $\lim_{n\to\infty} \mu(E_n) = \mu(E)$.

Thus, a measure is continuous from above and from below at every set in its ring of definition.

Next, we will formulate a result known as the "Approximation Theorem", which essentially states that in the case of a σ-finite measure on a ring $\mathcal{R}$, any set of finite measure in $\mathcal{S}(\mathcal{R})$ may be approximated "in measure" by a set of $\mathcal{R}$.

Theorem 4.4.4 *Let $\mathcal{R}$ be a ring and μ be a σ-finite measure on $\mathcal{S}(\mathcal{R})$. Then $\forall E \in \mathcal{S}(\mathcal{R})$ with $\mu(E) < \infty$, and $\forall \epsilon > 0$ $\exists F \in \mathcal{R}$ such that $\mu(E \Delta F) < \epsilon$.*

Note that, from the approximation theorem, it follows that any Borel set $B \in \mathcal{B}_{\mathbb{R}} = \mathcal{S}(\mathcal{R}(\mathcal{P}))$, where $\mathcal{P} = \{(a, b] : -\infty < a \leq b < \infty\}$, with finite Lebesgue measure, can be approximated (in the sense of the Lebesgue measure) by a finite union of disjoint intervals $(a, b]$.

4.5 Probability Spaces

Definition 4.5.1 The triple $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a σ-field of subsets of Ω, μ is a measure on $\mathcal{F}$, is called a *measure space*. The pair $(\Omega, \mathcal{F})$ is usually referred to as a *measurable space*, while any set $E \in \mathcal{F}$ is called *measurable* (or $\mathcal{F}$*-measurable* if there is any possible ambiguity).

Definition 4.5.2 A measure space $(\Omega, \mathcal{F}, P)$, where P is a probability measure on $\mathcal{F}$, is called a *probability space*, elements of $\mathcal{F}$ are called *events*, Ω is called a *sample space*, points in Ω are called *elementary outcomes*.

Examples of probability spaces:

1. $([0,1), \{\emptyset, [0,1), [0,1/3), [1/3,1)\}, \lambda)$, where λ is the length of an interval. This could be used as a probability model for tossing a biased coin once, where 'tail' is twice more likely to occur than 'head', with 'head'=[0,1/3) and 'tail'=[1/3,1).

2. Probability model for n tosses of a fair coin. The simplest probability model for n tosses of a fair coin is given by $(\Omega, \mathcal{F}, P)$, where
$\Omega = \{\omega_i = (x_{i,1}, \dots, x_{i,n}) : x_{i,k} \in \{0,1\}, k = 1, \dots, n; i = 1, \dots, 2^n\}$, $\mathcal{F} = 2^\Omega$ and the probability measure on $\mathcal{F}$ is defined by

$$P(E) = \frac{1}{2^n} \sum_{i=1}^{2^n} I_E(\omega_i), \quad \forall E \in \mathcal{F}.$$

(Here $(0,0,1,0,1,1,1)$ denotes the event that in 7 tosses, first, second and fourth tosses were "tails", while the rest were "heads".)

3. Probability model for an **infinite sequence** of coin tosses. In order to introduce a probability model for an infinite sequence of coin tosses it is convenient to consider binary digit expansions:

Each $0 \le \omega < 1$ may be written as

$$\omega = \sum_{i=1}^{\infty} \frac{\delta_i(\omega)}{2^i},$$

where, for every $i \in \mathbb{N}$,

$$\delta_i(\omega) = \left[2^i (\omega - \sum_{j=1}^{i-1} \frac{\delta_j(\omega)}{2^j}) \right] = 0 \text{ or } 1.$$

Here $[x]$ denotes the integer part of x (or the greatest integer which is less than or equal to x).
Proof of binary expansion: Note that $\delta_1(\omega) = [2\omega] = 0$ for $0 \le \omega < 1/2$ and $\delta_1(\omega) = 1$ for $1/2 \le \omega < 1$, and $0 \le \omega - \frac{1}{2}\delta_1(\omega) < \frac{1}{2}$ for all $\omega \in [0,1)$. Let

$$\omega_n = \sum_{j=1}^{n} \frac{\delta_j(\omega)}{2^j},$$

which is the nth binary approximation to ω. Suppose inductively that $\delta_j(\omega) = 0$ or 1 for $j = 1, \ldots, k$ and that

$$0 \le \omega - \omega_k < \frac{1}{2^k}$$

for some $k \ge 1$. Then

$$\delta_{k+1}(\omega) = \left[2^{k+1}(\omega - \omega_k)\right] = \begin{cases} 0, & \text{if } \ 0 \le \omega - \omega_k < \dfrac{1}{2^{k+1}}, \\ 1, & \text{if } \ \dfrac{1}{2^{k+1}} \le \omega - \omega_k < \dfrac{1}{2^k}. \end{cases}$$

$$0 \le \omega - \omega_{k+1} = \omega - \omega_k - \frac{\delta_{k+1}}{2^{k+1}} < \frac{1}{2^{k+1}}.$$

Thus, by induction, for all $n \ge 1$,

$$0 \le \omega - \omega_n < \frac{1}{2^n}.$$

Taking $n \to \infty$, we obtain that $\omega = \lim_{n\to\infty} \omega_n$.

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Take a sample space $\Omega = [0, 1)$ and a field

$$\mathcal{F} = \{[a_1, b_1) \cup \cdots \cup [a_m, b_m) : [a_i, b_i) \cap [a_j, b_j) = \emptyset \ \ \forall i \ne j, \ m \in \mathbb{N}, \ 0 \le a_i \le b_i \le 1 \ \ \forall i\}.$$

Regard the set $\{\omega \in \Omega : \delta_j(\omega) = 1\}$ as an event "head on jth toss". Note that

$$\{\omega \in \Omega : \delta_1(\omega) = 1\} = [\frac{1}{2}, 1) \in \mathcal{F},$$

$$\{\omega \in \Omega : \delta_2(\omega) = 1\} = [\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1) \in \mathcal{F},$$

$$\{\omega \in \Omega : \delta_3(\omega) = 1\} = [\frac{1}{8}, \frac{1}{4}) \cup [\frac{3}{8}, \frac{1}{2}) \cup [\frac{5}{8}, \frac{3}{4}) \cup [\frac{7}{8}, 1) \in \mathcal{F}, \ldots$$

Naturally, $\{\omega \in \Omega : \delta_j(\omega) = 0\}$ is interpreted as "tail on jth toss".
If $C = [a_1, b_1) \cup [a_2, b_2) \cup \cdots \cup [a_m, b_m) \in \mathcal{F}$, then define

$$P(C) = \sum_{i=1}^{m} (b_i - a_i) = \int_0^1 I_C(\omega) d\omega,$$

where the indicator $I_C(\omega) = 1$ if $\omega \in C$, and $I_C(\omega) = 0$ if $\omega \notin C$.

Proposition 4.5.1 *For $i_1, \ldots, i_n = 0$ or 1,*

$$P\{\omega \in \Omega : \delta_1(\omega) = i_1, \ldots, \delta_n(\omega) = i_n\} = \frac{1}{2^n}.$$

Proof: Note that

$$\{\omega \in \Omega : \delta_1(\omega) = i_1, \ldots, \delta_n(\omega) = i_n\} = \{\omega \in \Omega : \sum_{k=1}^{n} \frac{i_k}{2^k} \le \omega < \sum_{k=1}^{n} \frac{i_k}{2^k} + \frac{1}{2^n}\},$$

which is an interval of length $1/2^n$.

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