Stat 620, Fall 2005

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4.1 Measure on a semiring. Extension to a ring

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Definition 4.5.1 Let $(\Omega, \mathcal{F}, P)$ $(\Omega \neq \emptyset)$ be a probability space. For all $A, B \in \mathcal{F}$ such that $P(A) > 0$ define a *conditional probability* of event B given A by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Proposition 4.5.1 *Denote $P_A(B) = P(B|A)$. Then P_A is a probability measure on $\mathcal{F}$.*

Definition 4.5.2 A collection $\{A_i\}_{i \in J}$ is called a *partition* of Ω if $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $\Omega = \bigcup_{i \in J} A_i$.

Proposition 4.5.2 *Let $\{A_i\}_{i \in J}$ be a countable partition of Ω for which $P(A_i) > 0$ for all $i \in J$. Then $\forall B \in \mathcal{F}$,*

$$P(B) = \sum_{i \in J} P(B|A_i)P(A_i),$$

and

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i \in J} P(B|A_i)P(A_i)}.$$

Definition 4.5.3 Events A and B are called *independent* iff $P(A \cap B) = P(A)P(B)$.

If $P(A) > 0$, this is equivalent to $P(B|A) = P(B)$.

Definition 4.5.4 Events $A_1, \ldots, A_n$ are said to be *pairwise independent* if A_i and A_j are independent $\forall i \neq j$, and *jointly (or mutually) independent* if

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \ldots P(A_{i_k})$$

for all $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $\forall k \in \mathbb{N}$. Finally, an infinite collection $\{A_i\}_{i \in J}$ is called mutually (respectively, pairwise) independent if every finite subcollection is.

Note: Often one simply says that a collection of events is independent, meaning joint independence.
Example: $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F} = 2^{\Omega}$, $P(\{\omega\}) = 1/4$, $\omega = 1, 2, 3, 4$. Let $A_i = \{i, 4\}$, $i = 1, 2, 3$. Then it's easy to see that A_1, A_2, A_3 are pairwise independent, but not jointly independent.

Definition 4.5.5 *Classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ of events are called independent iff $\forall C_i \in \mathcal{C}_i$, $i = 1, \ldots, n$,*

$$P(C_1 \cap \cdots \cap C_n) = P(C_1) \ldots P(C_n).$$

Note that $C_1, \ldots, C_n$ are independent $\Leftrightarrow \mathcal{C}_1, \ldots, \mathcal{C}_n$, where $\mathcal{C}_i = \{C_i, \Omega\}$, are independent classes.

Theorem 4.5.1 *If $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are independent π-systems, then $\mathcal{A}_i = \sigma(\mathcal{C}_i)$, $i = 1, \ldots, n$, are independent σ-fields.*

Corollary 1: *$A_1, \ldots, A_n$ are mutually independent iff $P(B_1 \cap \cdots \cap B_n) = P(B_1) \ldots P(B_n)$ for all choices of $B_k = A_k$ or A_k^c, $k = 1, \ldots, n$.*

Corollary 2: *Let $\{A_i\}_{i \in J}$ be a countable collection of independent events, and let J_k, $k \in K$, be mutually exclusive non-empty subsets of J. Define*

$$\mathcal{A}_k = \sigma\left(\{A_j : j \in J_k\}\right), \quad k \in K.$$

Then $\{\mathcal{A}_k\}_{k \in K}$ are independent.

Example: If $A_1, A_2, \ldots$ are independent events, then, by Corollary 2, events

$$B = \bigcup_{k=1}^{\infty} A_{2k-1} \text{ and } C = \bigcup_{k=1}^{\infty} A_{2k}$$

are independent.
Example: Let $J = \{(i, j) : i \in \mathbb{N}, j \in \mathbb{N}\}$ and $J_k = \{(k, j) : j \in \mathbb{N}\}$ for $k = 1, 2 \ldots$. Suppose that the array

$$\begin{array}{ccc} A_{(1,1)}, & A_{(1,2)}, & \ldots \\ A_{(2,1)}, & A_{(2,2)}, & \ldots \\ \ldots & \ldots & \ldots \end{array}$$

of events is independent. If $\mathcal{A}_k$ is the σ-field generated by the kth row, i.e. $\mathcal{A}_k = \sigma(\{A_n : n \in J_k\})$, then, by Corollary 2, $\{\mathcal{A}_k\}_{k=1}^{\infty}$ is independent.

Proposition 4.5.3 *For any sequence of events* $\{A_n\}_{n\geq 1}$,

$$P(\underline{\lim} A_n) \leq \liminf_n P(A_n) \leq \limsup_n P(A_n) \leq P(\overline{\lim} A_n).$$

Proposition 4.5.4 *If* $0 \leq a_1, a_2 \cdots \leq 1$ *are nonnegative real numbers for which* $a_1 + a_2 + \cdots = \infty$, *then* $\lim_{n\to\infty}(1-a_1)(1-a_2)\ldots(1-a_n) = 0$.

Proof: Since $1 + x \leq e^x \ \forall x \in \mathbb{R}$, then

$$(1-a_1)(1-a_2)\ldots(1-a_n) \leq e^{-(a_1+\cdots+a_n)} \to 0 \ \text{ as } \ n \to \infty.$$

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Theorem 4.5.2 *(Borel-Cantelli Lemmas) If* $A_1, A_2, \ldots$ *is* **any** *sequence of events for which*

$$\sum_{n=1}^{\infty} P(A_n) < \infty,$$

then

$$P(A_n, i.o.) \equiv P(\overline{\lim} A_n) = 0.$$

Conversely, if $A_1, A_2, \ldots$ *are* **independent** and

$$\sum_{n=1}^{\infty} P(A_n) = \infty,$$

then

$$P(A_n, i.o.) \equiv P(\overline{\lim} A_n) = 1.$$

Example: An Application of Borel-Cantelli lemmas.
Let $\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}_{[0,1)}$ and P be the Lebesgue measure on $\mathcal{F}$. Recall that $\forall \omega \in \Omega$,

$$\omega = \sum_{k=1}^{\infty} \frac{\delta_k}{2^k},$$

where $\delta_k \in \{0,1\}$ is the kth digit in the binary expansion of ω. Let

$$A_k = \{\omega \in \Omega : \delta_k(\omega) = 0\}, \ \ k \geq 1.$$

Then $A_1, A_2, \ldots$ are independent, since for arbitrary $i_1 < i_2 < \cdots < i_n$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}) = P\left(\{\omega \in \Omega : \delta_{i_1}(\omega) = \delta_{i_2}(\omega) = \cdots = \delta_{i_n}(\omega) = 0\}\right)$$

$$= 2^{i_n - n} P\left(\{\omega \in \Omega : \delta_{i_1}(\omega) = \cdots = \delta_{i_n}(\omega), \delta_j(\omega) = 1, \forall j \in \{1, \ldots, i_n\} \setminus \{i_1, \ldots, i_n\}\}\right)$$

$$= \frac{2^{(i_n - n)}}{2^{i_n}} = \frac{1}{2^n} = P(A_{i_1}) \ldots P(A_{i_n}),$$

where $P(A_k) = \frac{1}{2}$ for each $k \geq 1$. Note that

$$\sum_{k=1}^{\infty} P(A_k) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty.$$

Thus, by Borel-Cantelli lemma, it follows that $P(A_k, i.o.) = 1$. (In an infinite sequence of tosses, there is with probability one an infinite number of tails.)
Let

$$L_n(\omega) = \begin{cases} 0, & \text{if } \delta_n(\omega) = 1, \\ k, & \text{if } \delta_n(\omega) = \cdots = \delta_{n+k-1}(\omega) = 0, \delta_{n+k}(\omega) = 1. \end{cases}$$

($L_n(\omega)$ is a length of a run of consecutive "tails" starting on the nth toss.) Then

$$P(\{\omega : L_n(\omega) \geq k\}) = P(A_n \cap \cdots \cap A_{n+k-1}) = \frac{1}{2^k}, \quad k \in \mathbb{N}.$$

Conjecture:

$$P(\{\omega : L_n(\omega) \geq (1+\epsilon)\log_2(n), i.o.\}) = \begin{cases} 0, & \text{if } \epsilon > 0 \\ 1, & \text{if } \epsilon = 0. \end{cases}$$

Proof: For $\epsilon > 0$,

$$P\left(\{\omega \in \Omega : L_n(\omega) \geq (1+\epsilon)\log_2(n)\}\right) = P\left(\{\omega \in \Omega : L_n(\omega) \geq \lceil (1+\epsilon)\log_2(n) \rceil\}\right)$$

$$= \frac{1}{2^{\lceil (1+\epsilon)\log_2(n) \rceil}} \leq \frac{1}{2^{(1+\epsilon)\log_2(n)}} = \left(2^{\log_2(n)}\right)^{-1-\epsilon} = \frac{1}{n^{1+\epsilon}}.$$

(Here $\lceil x \rceil$ is the smallest integer, which is greater than or equal to x. Also let us denote by $\lfloor x \rfloor$ the largest integer which is less than or equal to x. Then $\lfloor x \rfloor \leq x \leq \lceil x \rceil$ and $\lceil x \rceil \leq \lfloor x \rfloor + 1$.)
For $\epsilon > 0$, it follows that,

$$\sum_{n=1}^{\infty} P\left(\{\omega \in \Omega : L_n(\omega) \geq (1+\epsilon)\log_2(n)\}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty,$$

which, by 1st Borel-Cantelli's lemma, implies that for $\epsilon > 0$,

$$P\left(\{\omega \in \Omega : L_n(\omega) \geq (1+\epsilon)\log_2(n),\ i.o.\}\right) = 0.$$

Now put $\epsilon = 0$. For all $n \in \mathbb{N}$,

$$P(\{\omega \in \Omega : L_n(\omega) \geq \log_2(n)\}) = P(\{\omega \in \Omega : L_n(\omega) \geq \lceil \log_2(n) \rceil\}) \geq P(\{\omega \in \Omega : L_n(\omega) \geq \lfloor \log_2(n) \rfloor + 1\})$$

$$= \frac{1}{2^{\lfloor \log_2(n) \rfloor + 1}} \geq \frac{1}{2} \cdot \frac{1}{2^{\log_2(n)}} = \frac{1}{2n}.$$

Let $n_1 = 2$, $n_{k+1} = n_k + \lceil \log_2(n_k) \rceil$ for $k \geq 1$. Define $C_k := \{\omega \in \Omega : L_{n_k}(\omega) \geq \log_2(n_k)\}$, $k \in \mathbb{N}$.

Then $(C_k)_{k=1}^{\infty}$ is a sequence of independent events, since $C_k = A_{n_k} \cap \cdots \cap A_{n_{k+1}-1}$. Moreover,

$$\sum_{k=1}^{\infty} P(C_k) = \sum_{k=1}^{\infty} P\left(\{\omega \in \Omega : L_{n_k}(\omega) \geq \log_2(n_k)\}\right) \geq \sum_{k=1}^{\infty} \frac{1}{2n_k},$$

where

$$\frac{1}{n_k} = \frac{n_{k+1} - n_k}{n_k \lceil \log_2(n_k) \rceil} = \sum_{j=n_k}^{n_{k+1}-1} \frac{1}{n_k \lceil \log_2(n_k) \rceil} \geq \sum_{j=n_k}^{n_{k+1}-1} \frac{1}{j \lceil \log_2(j) \rceil},$$

since for all $j \geq n_k$, we have that $\dfrac{1}{n_k} \geq \dfrac{1}{j}$ and $\dfrac{1}{\lceil \log_2(n_k) \rceil} \geq \dfrac{1}{\lceil \log_2(j) \rceil}$. Therefore,

$$\sum_{k=1}^{\infty} P(C_k) \geq \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{n_k} \geq \frac{1}{2} \sum_{k=1}^{\infty} \sum_{j=n_k}^{n_{k+1}-1} \frac{1}{j \lceil \log_2(j) \rceil} = \frac{1}{2} \sum_{j=2}^{\infty} \frac{1}{j \lceil \log_2(j) \rceil} = \infty.$$

Thus, noting that $\{L_n \geq \log_2(n), i.o.\} \supset \{C_n, i.o.\}$ and applying 2nd Borel-Cantelli's lemma to independent events $\{C_n\}$, we obtain that

$$P\left(\{\omega \in \Omega : L_n(\omega) \geq \log_2(n), i.o.\}\right) \geq P(C_n, i.o.) = 1.$$

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4.6 Tail σ-field. Kolmogorov's 0-1 law

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{A}_i$, $i \in \mathbb{N}$, be the σ-fields of subsets of Ω such that $\mathcal{A}_i \subset \mathcal{F}$. Define

$$\mathcal{C}_n := \sigma\left(\bigcup_{i \geq n} \mathcal{A}_i\right)$$

and

$$\mathcal{C}_\infty := \bigcap_{n=1}^{\infty} \mathcal{C}_n.$$

Definition 4.6.1 $\mathcal{C}_\infty$ is called the *tail σ-field generated by* $(\mathcal{A}_i)_{i \in \mathbb{N}}$.

Example: Let $\mathcal{A}_i = \{A_i, A_i^c, \Omega, \emptyset\}$, $\mathcal{C}_n = \sigma\left(\bigcup_{i=n}^{\infty} \mathcal{A}_i\right)$. Then

$$\overline{\lim} A_k = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \bigcap_{n=k}^{\infty} (\bigcup_{m=n}^{\infty} A_m) \in \mathcal{C}_k \ \ \forall k \in \mathbb{N},$$

since for all $n \geq k$, $\bigcup_{m=n}^{\infty} A_m \in \sigma(\bigcup_{i=n}^{\infty} \mathcal{A}_i) \subset \sigma(\bigcup_{i=k}^{\infty} \mathcal{A}_i)$. Thus,

$$\overline{\lim} A_k \in \bigcap_{k=1}^{\infty} \mathcal{C}_k = \mathcal{C}_\infty.$$

Theorem 4.6.1 *(Kolmogorov's 0-1 Law) If $\mathcal{A}_i$, $i \in \mathbb{N}$, are independent, then for every $C \in \mathcal{C}_\infty$, $P(C) = 0$ or 1.*