Stat 620, Fall 2005

October 11, 2005

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5 Measurable functions and transformations

5.1 Properties of Measurable Transformations

Let T be a transformation from a subset D of a space Ω into a space Ω'. (That is $T : D \longrightarrow \Omega'$.)

Definition 5.1 The *inverse image* $T^{-1}G$ of a subset $G \subset \Omega'$ is defined by:

$$T^{-1}G := \{\omega \in D : T(\omega) \in G\}.$$

Proposition 5.1 *Let $T : D \longrightarrow \Omega'$, where $D \subset \Omega$. Let G, H, G_i, $i = 1, 2, \ldots,$ be subsets of Ω'. Then*

$$(i)\ \ T^{-1}(G - H) = (T^{-1}G) - (T^{-1}H),$$

$$(ii)\ \ T^{-1}\left(\bigcup_{i=1}^{\infty} G_i\right) = \bigcup_{i=1}^{\infty}(T^{-1}G_i),\ \ (iii)\ \ T^{-1}\left(\bigcap_{i=1}^{\infty} G_i\right) = \bigcap_{i=1}^{\infty}(T^{-1}G_i),$$

$$(iv)\ \ T^{-1}(G^c) = D - (T^{-1}G).\ \textit{In particular, if } D = \Omega, \textit{ then } T^{-1}(G^c) = (T^{-1}G)^c.$$

Notation: Let $\mathcal{G}$ be a class of subsets of Ω'. Then

$$T^{-1}\mathcal{G} := \{T^{-1}G : G \in \mathcal{G}\}.$$

Theorem 5.1 *Let T be a transformation from $D \subset \Omega$ into Ω'. Let $\mathcal{S}$ be a σ-ring of subsets of Ω'. Then $T^{-1}\mathcal{S}$ is a σ-ring in Ω. (If $D = \Omega$ and $\mathcal{S}$ is a σ-field of subsets of Ω', then $T^{-1}\mathcal{S}$ is a σ-field of subsets of Ω.)*

Theorem 5.2 *: Let $(\Omega, \mathcal{F})$ be a measurable space and let T be a transformation from a subset of Ω into Ω'. Then the class*

$$\mathcal{G} := \{G \subset \Omega' : (T^{-1}G) \in \mathcal{F}\} \ \ is \ \ a \ \sigma-ring.$$

Definition 5.2 Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measurable spaces, and T is a transformation from a subset of Ω into Ω'. Then T is called *$\mathcal{F}/\mathcal{F}'$-measurable* (or, simply, *measurable,* if there is no possibility of confusion) iff $T^{-1}(\mathcal{F}') \subset \mathcal{F}$, i.e. $(T^{-1}G) \in \mathcal{F}$ whenever $G \in \mathcal{F}'$.

Remark: Note that the domain D of definition of a measurable transformation must be a measurable set. Namely, let $T : D \to \Omega'$ be a $\mathcal{F}/\mathcal{F}'$-measurable transformation, then $D = (T^{-1}\Omega') \in \mathcal{F}$.

Theorem 5.3 *: Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$, $(\Omega_3, \mathcal{F}_3)$ be measurable spaces. Let T_1 be an $\mathcal{F}_1/\mathcal{F}_2$-measurable transformation from $D_1 \subset \Omega_1$ into Ω_2, T_2 be an $\mathcal{F}_2/\mathcal{F}_3$-measurable transformation from $D_2 \subset \Omega_2$ into Ω_3. Then the composition $(T_2 \circ T_1)(\omega) := T_2(T_1(\omega))$, defined on all $\omega \in D_1$ such that $T_1(\omega) \in D_2$, is an $\mathcal{F}_1/\mathcal{F}_3$-measurable transformation.*

Theorem 5.4 *: Let $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ be measurable spaces, and $\mathcal{C}$ is a class of subsets of Ω' such that $\mathcal{S}(\mathcal{C}) = \mathcal{F}'$. Let T be a transformation from a subset of Ω into Ω'. Then T is $\mathcal{F}/\mathcal{F}'$-measurable iff $(T^{-1}G) \in \mathcal{F}$ for all $G \in \mathcal{C}$.*

5.2 Measures Induced by Transformations

Theorem 5.5 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $(\Omega', \mathcal{F}')$ be a measurable space, and T be a measurable transformation from a subset D of Ω into Ω'. Then the set function μT^{-1} defined on $\mathcal{F}'$ by*

$$(\mu T^{-1})(G) := \mu(T^{-1}G), \ \ \forall G \in \mathcal{F}'$$

is a measure on $\mathcal{F}'$.

Definition 5.3 μT^{-1} is called a *measure induced* on $\mathcal{F}'$ from μ on $\mathcal{F}$ by a measurable transformation T.

5.3 Measurable Functions

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and $\overline{\mathcal{B}}_{\mathbb{R}} := \{B, B \cup \{+\infty\}, B \cup \{-\infty\}, B \cup \{-\infty\} \cup \{+\infty\} : B \in \mathcal{B}_{\mathbb{R}}\}$. It's easy to see that $\overline{\mathcal{B}}_{\mathbb{R}}$ is a σ-field. $\overline{\mathcal{B}}_{\mathbb{R}}$ is called the *extended Borel σ-field.*

Definition 5.4 Let $(\Omega, \mathcal{F})$ and $(\overline{\mathbb{R}}, \overline{\mathcal{B}}_{\mathbb{R}})$ be measurable spaces and T be a transformation from a subset of Ω into $\overline{\mathbb{R}}$. If T is $\mathcal{F}/\overline{\mathcal{B}}_{\mathbb{R}}$-measurable, then T is called a *measurable* (or *$\mathcal{F}$-measurable*) *function.* If in the above definition, $(\overline{\mathbb{R}}, \overline{\mathcal{B}}_{\mathbb{R}})$ is replaced with $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, T is called a *Borel measurable function.*

Proposition 5.2 $\overline{\mathcal{B}}_{\mathbb{R}} = \mathcal{S}(\{\{\infty\}, \{-\infty\}, (-\infty, a] : -\infty < a < \infty\})$ *and*

$$\overline{\mathcal{B}}_{\mathbb{R}} = \mathcal{S}(\{\overline{\mathbb{R}}, [-\infty, a] : -\infty < a < \infty\}).$$

Application of theorem 5.4 gives the following criteria for measurability of functions:

Theorem 5.6 *Let $(\Omega, \mathcal{F})$ be a measurable space, f be a function defined on a subset $D \subset \Omega$. Then the following conditions are equivalent:*
(i) f is measurable;
(ii) $f^{-1}\{\infty\} \in \mathcal{F}$, $f^{-1}\{-\infty\} \in \mathcal{F}$ and either:
(a) $f^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathcal{B}_{\mathbb{R}}$, or (b) $f^{-1}(-\infty, a] \in \mathcal{F}$ $\forall a \in \mathbb{R}$;
(iii) $D \in \mathcal{F}$ and $\{\omega \in D : f(\omega) \leq a\} = f^{-1}[-\infty, a] \in \mathcal{F}$ for all $a \in \mathbb{R}$.

Note: The set $(-\infty, a]$ in part (b) of the above theorem may be replaced by $(-\infty, a)$ or $[a, \infty)$ or (a, ∞) and "real a" may be replaced by "rational a". Similarly, the set $[-\infty, a]$ in part (iii) may be replaced by $[-\infty, a)$ or $[a, \infty]$ or $(a, \infty]$ and "real a" may be replaced by "rational a". Condition $D \in \mathcal{F}$ is important in (iii) since $f^{-1}[-\infty, \infty] = D$.

Example: The simplest example of a measurable function on $(\Omega, \mathcal{F})$ is given by $\chi_E \equiv I_E$ (indicator) of a set E, where $E \in \mathcal{F}$.

Definition 5.5 For any extended real function f defined on a subset of Ω, put

$$f_+(\omega) := \max\{0, f(\omega)\}, \quad f_-(\omega) := -\min\{0, f(\omega)\} \ \ \forall \omega.$$

Functions f_+, f_- are called *positive*, *negative part* of f respectively. Note that $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Let us fix a measurable space $(\Omega, \mathcal{F})$ for the remainder of this section.

Theorem 5.7 *If f, g are measurable functions and $c \in \mathbb{R}$, then cf, $f + g$, fg, f/g,* $\max(f, g)$, $\min(f, g)$, *f_+, f_-, $|f|$ are all measurable functions.*

Theorem 5.8 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on Ω. Then*

$$\sup_n f_n, \quad \inf_n f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n$$

are all measurable.

Theorem 5.9 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on Ω. Let E be a set of all ω for which $f_n(\omega)$ converges (to a finite or infinite value). Then E is a measurable set and $f(\omega) := \lim_{n\to\infty} f_n(\omega)$, defined on E, is a measurable function.*

5.4 Random Variables

Definition 5.6 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space. Suppose $T : \Omega \to E$ and T is $\mathcal{F}/\mathcal{E}$-measurable, then T is called an *E-valued random variable.* Then the induced measure $\mu_T := PT^{-1}$ on $\mathcal{E}$ is a *probability* measure on $\mathcal{E}$, since

$$(PT^{-1})(E) = P(T^{-1}E) = P(\{\omega \in \Omega : T(\omega) \in E\}) = P(\Omega) = 1.$$

(PT^{-1}) is called a *probability law* (or, a *probability distribution*) of an E-valued random variable T.

Definition 5.7 If $(E, \mathcal{E}) = (\overline{\mathbb{R}}, \overline{\mathcal{B}}_{\mathbb{R}})$, an E-valued random variable is called an *extended real random variable.* In the case of $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, an E-valued random variable is called a *(real) random variable.*

Examples: 1. Take $\Omega = [-1, 1]$, $\mathcal{F} = \mathcal{B}_{[-1,1]} \equiv \mathcal{B}_{\mathbb{R}} \cap [-1, 1]$, $P = \frac{1}{2}\lambda$, where λ is the Lebesgue measure on $\mathcal{F}$. Then $(\Omega, \mathcal{F}, P)$ is a probability space. Let $X(\omega) = |\omega|$ for all $\omega \in \Omega$. Then X is a random variable and it's probability law on $\mathcal{B}_{\mathbb{R}}$ is given by:

$$\mu_X(B) = (PX^{-1})(B) = P(X^{-1}B) = P(\{\omega \in \Omega : X(\omega) \in B\}) = \frac{1}{2}\lambda(\{\omega \in [-1, 1] : |\omega| \in B\})$$

for all $B \in \mathcal{B}_{\mathbb{R}}$. In particular, for $B = (-\infty, x]$,

$$\mu_X(-\infty, x] = \begin{cases} 0, & \text{if } x \notin [0, 1], \\ x, & \text{if } x \in [0, 1], \end{cases}$$

which is a Uniform probability law on $[0, 1]$.
2. Take $\Omega = [-1, 1]$, $\mathcal{F} = \mathcal{B}_{[-1,1]} \equiv \mathcal{B}_{\mathbb{R}} \cap [-1, 1]$ (as before), but now let $P = \frac{1}{2}\left(\delta_{\{-1\}} + \delta_{\{1\}}\right)$, where $\forall F \in \mathcal{F}$,

$$\delta_{\{c\}}(F) = \begin{cases} 0, & \text{if } c \notin F, \\ 1, & \text{if } c \in F. \end{cases}$$

Then $(\Omega, \mathcal{F}, P)$ is a probability space. Let $X(\omega) = |\omega|$ for all $\omega \in \Omega$. Then X is a random variable and its probability law on $\mathcal{B}_{\mathbb{R}}$ is given by:

$$\mu_X(B) = P(X^{-1}B) = P(\{\omega \in \Omega : X(\omega) \in B\}) = \frac{1}{2}(\delta_{\{-1\}}+\delta_{\{1\}})(\{\omega \in [-1,1] : |\omega| \in B\}) = \delta_{\{1\}}(B)$$

for all $B \in \mathcal{B}_{\mathbb{R}}$. Note that $P(X = 1) \equiv P(\{\omega \in \Omega : X(\omega) = 1\}) = 1$ in this case.

5.5 Simple Functions and Characterization Theorem

Definition 5.8 Let $(\Omega, \mathcal{F})$ be a measurable space. If $F_1, \ldots, F_k \in \mathcal{F}$ are mutually exclusive subsets of Ω and $c_1, \ldots, c_k \in \overline{\mathbb{R}}$, then a function f of the form:

$$f = c_1 I_{F_1} + \cdots + c_k I_{F_k}$$

is called *simple* (or, $\mathcal{F}$*-simple*). (Note that simple functions are measurable.)

Theorem 5.10 *(Characterization Theorem): If $(\Omega, \mathcal{F})$ is a measurable space, then a function $f : \Omega \to \overline{\mathbb{R}}$ is measurable iff $\exists$ $\mathcal{F}$-simple functions $g_1, g_2, \ldots$ for which*

$$\lim_{n\to\infty} g_n(\omega) = f(\omega)$$

for all $\omega \in \Omega$.

Corollary 1: If $f \geq 0$, then $\{g_n\}_{n=1}^{\infty}$ is an increasing sequence of non-negative simple functions such that $g_n(\omega) \uparrow f(\omega)$ for all $\omega \in \Omega$.

Corollary 2: If f is bounded, then $\lim_{n\to\infty} \sup_{\omega\in\Omega} |g_n(\omega) - f(\omega)| = 0$.