Stat 620, Fall 2005

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1 Real line and its subsets

2 Topology and metric spaces

3 Classes of sets

4 Measures

5 Measurable functions and transformations

6 Lebesgue integral

6.1 Lebesgue integral of a nonnegative simple function

Let $\mathbb{S}_+$ be the class of nonnegative simple functions on a measure space $(\Omega, \mathcal{F}, \mu)$, i.e. the class of functions f of the form:

$$f = c_1 I_{F_1} + \cdots + c_k I_{F_k},$$

where $c_1, \cdots, c_k \in [0, +\infty]$, and $F_1, \cdots, F_k \in \mathcal{F}$ form a finite partition of Ω.

Definition 6.1 *For an arbitrary $f = \sum_{i=1}^k c_i I_{F_i} \in \mathbb{S}_+$, define*

$$\int f d\mu \equiv \int_\Omega f d\mu := \sum_{i=1}^k c_i \mu(F_i).$$

Note: In the above definition and in the remainder of the section, we set the convention $\infty(0) = 0$. The Lebesgue integral for class $\mathbb{S}_+$ is well-defined, since: 1) $c_i \geq 0$ (note that μ need not be finite); and 2) If there is a second representation $f = a_1 I_{G_1} + \cdots + a_m I_{G_m}$, then for any pair (i, j), either $F_i \cap G_j = \emptyset$ or else $c_i = a_j$, thus,

$$\sum_{i=1}^k c_i \mu(F_i) = \sum_{i=1}^k c_i \mu(F_i \cap (\cup_{j=1}^m G_j)) = \sum_{i=1}^k \sum_{j=1}^m c_i \mu(F_i \cap G_j) = \sum_{j=1}^m \sum_{i=1}^k a_j \mu(F_i \cap G_j) = \sum_{j=1}^m a_j \mu(G_j).$$

Theorem 6.1 *(i) For arbitrary $f, g \in \mathbb{S}_+$, and $a, b \in \mathbb{R}_+$,*

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu.$$

(ii) For all $f, g \in \mathbb{S}_+$ *such that* $f(\omega) \geq g(\omega)$ *for all* $\omega \in \Omega$,

$$\int f d\mu \geq \int g d\mu.$$

(iii) If $(f_n)_{n=1}^{\infty}$ *is an increasing sequence in* $\mathbb{S}_+$ *and* $f(\omega) := \lim_{n\to\infty} f_n(\omega) \geq g(\omega)$ *for all* $\omega \in \Omega$, *where* $g \in \mathbb{S}_+$, *then*

$$\lim_{n\to\infty} \int f_n d\mu \geq \int g d\mu.$$

6.2 Lebesgue integral of a nonnegative measurable function

Definition 6.2 *By characterization theorem, for every nonnegative measurable function* f *defined on* Ω, *there exists an increasing sequence of functions* $f_n \in \mathbb{S}_+$, *such that* $f_n(\omega) \uparrow f(\omega)$ *for all* $\omega \in \Omega$. *Thus, define*

$$\int f d\mu := \lim_{n\to\infty} \int f_n d\mu.$$

Note: In the above definition, the limit exists due to property (*ii*) of Theorem 6.1. The integral will be well-defined once we show that the value does not depend on the choice of the approximating sequence.

Definition 6.3 *For a measurable* $f \geq 0$ *defined on* Ω *and an arbitrary* $E \in \mathcal{F}$, *let*

$$\int_E f d\mu := \int f I_E \, d\mu.$$

This set function is called the indefinite integral of f.

Theorem 6.2 *Let* $f, g \geq 0$ *be measurable functions on* Ω.
(i) For all $a, b \geq 0$,

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu.$$

(ii) If $f(\omega) \geq g(\omega)$ *for all* $\omega \in \Omega$, *then*

$$\int f d\mu \geq \int g d\mu.$$

(iii) If $f \geq 0$ *is a measurable function on* Ω *and* $E \in \mathcal{F}$ *such that* $\mu(E) = 0$, *then*

$$\int_E f d\mu = 0.$$

6.3 Lebesgue integral of more general classes of measurable functions

Definition 6.4 *Consider measurable functions* $f = f_+ - f_-$ *defined on* Ω*, with* $f_+ := \max(f, 0)$ *and* $f_- := -\min(f, 0)$*, such that both* $\int f_+ d\mu < \infty$ *and* $\int f_- d\mu < \infty$*. Such functions* f *are called integrable and the Lebesgue integral of* f *is defined as:*

$$\int f d\mu := \int f_+ d\mu - \int f_- d\mu.$$

The indefinite integral of integrable f*, for* $E \in \mathcal{F}$*, is then defined by* $\int_E f d\mu := \int f I_E \, d\mu$*.*

Definition 6.5 *Consider measurable functions* $f = f_+ - f_-$ *on* Ω*, such that* $\int f_+ d\mu < \infty$ *but* $\int f_- d\mu = \infty$*, then define* $\int f d\mu := -\infty$*. On the other hand, for a measurable function* $f = f_+ - f_-$ *on* Ω*, such that* $\int f_- d\mu < \infty$ *but* $\int f_+ d\mu = \infty$*, define* $\int f d\mu := \infty$*.*

Definition 6.6 *We say that a given property holds almost everywhere with respect to* μ *(a.e.*(μ)*) on* Ω*, if the set of* ω*'s where that property fails has measure* μ *equal to 0.*

Proposition 6.1 *Let* f *be integrable function on* Ω*.*
(i) If E, F *are disjoint measurable sets, then* $\int_{E \cup F} f d\mu = \int_E f d\mu + \int_F f d\mu$*.*
(ii) For $E \in \mathcal{F}$ *with* $\mu(E) = 0$*, it follows that* $\int_E f d\mu = 0$*.*
(iii) If g *is another measurable function and* $f = g$ *a.e.*(μ)*, then* $\int f d\mu = \int g d\mu$*.*

Since value of $\int f d\mu$ is not affected by values of f on a set of measure zero, it is natural to extend the class of integrands to functions defined almost everywhere on Ω.

Definition 6.7 *Let* $L_1 = L_1(\Omega, \mathcal{F}, \mu)$ *be the class of all measurable functions* f *defined almost everywhere on* Ω *and such that* $f = g$ *a.e. for some integrable function* g*. Then* $\int f d\mu := \int g d\mu$*.*

Similarly, one extends definition of Lebesgue integral to integrands defined almost everywhere on Ω in the case of functions from Definition 6.5.

Theorem 6.3 *(i) If* $f \in L_1$ *then* f *is finite a.e.*
(ii) If $f \in L_1$ *then* $\exists$ *finite-valued integrable* g *such that* $f = g$ *a.e.*
(iii) For all $f, g \in L_1$ *and* $\forall a, b \in \mathbb{R}$*, it follows that* $(af + bg) \in L_1$ *and* $\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$*.*
(iv) If f *is a measurable function such that* $\int f d\mu$ *is defined, and* $E \in \mathcal{F}$ *with* $\mu(E) = 0$*, then* $\int_E f d\mu = 0$*.*
(v) If $f \in L_1$*,* g *is measurable and* $f = g$ *a.e., then* $g \in L_1$ *and* $\int f d\mu = \int g d\mu$*.*
(vi) If $f \notin L_1$ *but* $\int f d\mu$ *is defined, and* g *is measurable with* $g = f$ *a.e., then* $g \notin L_1$ *but* $\int g d\mu$ *is defined and* $\int g d\mu = \int f d\mu$ $(= \pm\infty)$*.*

Theorem 6.4 *(i) For f, g measurable functions defined a.e., with $f \geq g$ a.e. and such that $\int f d\mu$ and $\int g d\mu$ are defined. Then $\int f d\mu \geq \int g d\mu$.*
(ii) Let $f \in L_1$ and g be a measurable function defined a.e. and suppose that $|g| \leq |f|$ a.e. Then $g \in L_1$.

Theorem 6.5 *Let f be a measurable function defined a.e. Then the following conditions are equivalent:*
(i) $f \in L_1$;
(ii) both $f_+ \in L_1$ and $f_- \in L_1$;
(iii) $|f| \in L_1$.
Moreover, if $f \in L_1$, then $\left|\int f d\mu\right| \leq \int |f| d\mu$.

Theorem 6.6 *(i) If f is a measurable function, $f \geq 0$ a.e. and $\int f d\mu = 0$, then $f = 0$ a.e.*
(ii) If $f \in L_1$ and $\int_E f d\mu = 0$ for all $E \in \mathcal{F}$, then $f = 0$ a.e.

Corollary: If $f, g \in L_1$ and $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{F}$, then $f = g$ a.e.

Theorem 6.7 *If $f \in L_1$, then $\mu\{\omega \in \Omega : |f(\omega)| \geq \epsilon\} < \infty$ for every $\epsilon > 0$ and the set $\{\omega \in \Omega : f(\omega) \neq 0\}$ has σ-finite measure.*

6.4 Convergence of integrals

Let $(\Omega, \mathcal{F}, \mu)$ be an underlying measure space.

Theorem 6.8 *(Monotone Convergence Theorem) Let $\{f_n\}$ be a sequence of a.e. nonnegative measurable functions each defined a.e. and such that $f_n(\omega) \leq f_{n+1}(\omega)$ a.e. for each $n \geq 1$. Let f be a measurable function defined and nonnegative a.e. on Ω, and such that $f_n(\omega) \uparrow f(\omega)$ a.e. Then*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Corollary 1: *For $f \in L_1$, its indefinite integral is an absolutely continuous set function, i.e. $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall E \in \mathcal{F}$ with $\mu(E) < \delta$, it follows that $\left|\int_E f d\mu\right| < \epsilon$.*

Corollary 2: *Let $\{f_n\}$ be a sequence of (a.e.) nonnegative measurable functions defined (a.e.) on Ω. Then $\sum_{n=1}^{\infty} f_n$ is an (a.e.) nonnegative measurable function on Ω and*

$$\int (\sum_{n=1}^{\infty} f_n) d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

Theorem 6.9 *(Fatou's Lemma) Let $\{f_n\}$ be a sequence of a.e. nonnegative measurable functions each defined a.e. on Ω. Then*

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int (\liminf_{n\to\infty} f_n) d\mu.$$

Theorem 6.10 *(Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of L_1-functions such that $|f_n| \leq |g|$ a.e. for some $g \in L_1$, for each $n = 1, 2, \ldots$. Let f be measurable and such that $f_n(\omega) \to f(\omega)$ a.e. Then $f \in L_1$ and*

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

Since $\left|\int f_n d\mu - \int f d\mu\right| = \left|\int (f_n - f) d\mu\right| \leq \int |f_n - f| d\mu$, it also follows that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

6.5 Transformation of integrals

Theorem 6.11 *(Transformation Theorem) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(\Omega', \mathcal{F}')$ be a measurable space and let T be a measurable transformation defined (a.e.) on Ω into Ω', and f a measurable function defined (a.e.) on Ω'. Then*

$$\int_{\Omega'} f d(\mu T^{-1}) = \int_{\Omega} (f \circ T) d\mu,$$

whenever f is nonnegative (a.e.), or (μT^{-1})-integrable, or $f \circ T$ is μ-integrable.

6.6 Some ramifications for probability theory

Let X be a r.v. on the probability space $(\Omega, \mathcal{F}, P)$. Let X_+, X_- be the positive and negative parts of X. If at least one of the integrals $\int_\Omega X_+(\omega)dP(\omega)$, $\int_\Omega X_-(\omega)dP(\omega)$ is finite, the integral $\int_\Omega X(\omega)dP(\omega)$ is well-defined and is called the expected value (or mean) of X, denoted by $E(X)$ or μ_X. In other words,

$$E(X) := \int_\Omega X(\omega)dP(\omega) \equiv \int_\Omega X(\omega)P(d\omega),$$

whenever the integral is well-defined. In particular, if $X \in L_1$ (or, equivalently, $|X| \in L_1$), then $E(X)$ is finite (and $E|X| < \infty$).

Moreover, by the transformation theorem, if f is a measurable function on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ and X is a random variable on $(\Omega, \mathcal{F}, P)$, and either $f \geq 0$ a.e. (μ_X), or $\int_\mathbb{R} |f| d\mu_X < \infty$, or $E|f(X)| < \infty$, where $\mu_X(B) = (PX^{-1})(B)$ is the probability measure induced on the Borel sets $B \in \mathcal{B}_\mathbb{R}$ by the random variable X, then

$$E(f(X)) \equiv \int_\Omega f(X(\omega))dP(\omega) = \int_\mathbb{R} f(x)d\mu_X(x).$$

Note also that theorems on convergence of integrals immediately give theorems for convergence of expectations for sequences of random variables. For example, if $(X_n)_{n\geq 1}$ is a monotone increasing sequence of nonnegative random variables in the probability space $(\Omega, \mathcal{F}, P)$ such that $X_n(\omega) \uparrow X(\omega)$ almost surely (i.e. almost everywhere, or with probability one) for some random variable X, then $E(X_n) \to E(X)$ as $n \to \infty$. (This is a "probabilistic" version of the monotone convergence theorem. Similar formulations can be obtained for the dominated convergence theorem and the Fatou's lemma.)

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6.7 Lebesgue integration on the real line

For all Borel measurable functions f, monotone convergence theorem gives that

$$\int_{\mathbb{R}} |f| d\lambda = \lim_{n\to\infty} \int_{[-n,n]} |f| d\lambda,$$

where λ is the Lebesgue measure on the Borel sets. Thus, in particular,

$$f \in L_1 \text{ iff } \lim_{n\to\infty} \int_{[-n,n]} |f| d\lambda < \infty.$$

In practice, one often deals with a function f which is Riemann integrable on every finite interval. We will show that it then follows that

$$\int_{[a,b]} f d\lambda = \int_a^b f(x)dx,$$

where the integral on the right is the (familiar from calculus) Riemann integral, and therefore

$$\int_{\mathbb{R}} f d\lambda = \lim_{n\to\infty} \int_{-n}^{n} f_+(x)dx - \lim_{n\to\infty} \int_{-n}^{n} f_-(x)dx,$$

if at least one of the two limits is finite.

The point is that it is easiest to evaluate the Lebesgue integral by Riemann procedure when possible. However, there are functions which are Lebesgue- but not Riemann-integrable on $[a, b]$.
Riemann integrability: Consider a finite interval $[a, b]$. Consider a finite partition $\mathcal{I} := \{a = x_0 < x_1 < \cdots < x_m = b\}$ of $[a, b]$. If f is a bounded function on $[a, b]$, say $|f(x)| \le M$ for all $a \le x \le b$, let

$$S_*(f, \mathcal{I}) := \sum_{j=1}^{m} \left[\inf_{x_{j-1}\le x\le x_j} f(x) \right] (x_j - x_{j-1})$$

and

$$S^*(f, \mathcal{I}) := \sum_{j=1}^{m} \left[\sup_{x_{j-1}\le x\le x_j} f(x) \right] (x_j - x_{j-1}).$$

Further let $S_*(f) := \sup_{\mathcal{I}} S_*(f, \mathcal{I})$ and $S^*(f) := \inf_{\mathcal{I}} S^*(f, \mathcal{I})$. Then f is said to be Riemann-integrable over $[a, b]$ iff $S_*(f) = S^*(f)$, in which case

$$\int_a^b f(x)dx := S_*(f) = S^*(f).$$

Theorem 6.12 *If f is Borel measurable and Riemann-integrable over $[a, b]$, then*

$$\int_a^b f(x)dx = \int_{[a,b]} f d\lambda.$$

Example (Computation of Lebesgue integral through Riemann integration):

$$\int_{[0,\infty)} e^{-x} d\lambda = \lim_{n\to\infty} \int_{[0,n]} e^{-x} d\lambda = \lim_{n\to\infty} \int_0^n e^{-x} dx = \lim_{n\to\infty} (1 - e^{-n}) = 1.$$

Example (Function which is Lebesgue integrable on $[0, 1]$ but not Riemann-integrable): Define function f on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then $f = 0$ a.e.(λ), thus $\int_{[0,1]} f d\lambda = 0$. However,

$$S_*(f, \mathcal{I}) = 0, \quad S^*(f, \mathcal{I}) = 1$$

for all finite partitions $\mathcal{I}$ of $[0, 1]$, thus $S_*(f) \neq S^*(f)$, implying that f is not Riemann-integrable on $[0, 1]$.

Note: In general, even if f is Riemann-integrable on every finite interval, the Lebesgue integral of f over $\mathbb{R}$ may fail to exist. Also, the existence of the limit $\lim_{n\to\infty} \int_{-n}^n f(x)dx$ does not imply existence of the Lebesgue integral $\int_\mathbb{R} f d\lambda$. For example, consider

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

Then f is Riemann-integrable on finite intervals and

$$\lim_{n\to\infty} \int_{-n}^n f(x)dx = \int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2},$$

but

$$\int_\mathbb{R} f_+(x)d\lambda = \int_{(0,+\infty)} \max\left\{0, \frac{\sin(x)}{x}\right\} d\lambda = +\infty,$$

and, similarly, $\int_\mathbb{R} f_-(x)d\lambda = +\infty$, implying that the Lebesgue integral of f does not exist. Indeed, note that for all $n = 0, 1, 2, \dots$ and $\forall x \in [2\pi n + \frac{\pi}{6}, 2\pi n + \frac{\pi}{2}]$,

$$\frac{\sin(x)}{x} \geq \frac{\frac{1}{2}}{2\pi n + \frac{\pi}{2}}.$$

Therefore,

$$\int_\mathbb{R} f_+ d\lambda \geq \int_{\bigcup_{n=0}^\infty [2\pi n + \frac{\pi}{6}, 2\pi n + \frac{\pi}{2}]} \frac{\sin(x)}{x} d\lambda = \sum_{n=0}^\infty \int_{[2\pi n + \frac{\pi}{6}, 2\pi n + \frac{\pi}{2}]} \frac{\sin(x)}{x} d\lambda \geq \frac{\pi}{3} \sum_{n=0}^\infty \frac{1/2}{2\pi n + \pi/2} = +\infty.$$