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Erratum

# Product formula and independence criterion for multiple Huang–Cambanis integrals

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**Abstract**

Multiple stochastic integrals of Huang and Cambanis [1978. Stochastic and multiple Wiener integrals for Gaussian processes. Ann. Probab. 6, 585–614] with respect to a general Gaussian process $X = (X_t, t \in \mathbb{T})$, whose covariance function is of bounded variation on bounded subsets of $\mathbb{T} \times \mathbb{T}$, are considered. A product formula for the integrals is derived and a necessary and sufficient condition for independence of multiple Huang–Cambanis integrals is obtained. As an illustration, the results are applied to the special case of multiple integrals with respect to a persistent fractional Brownian motion.


*MSC:* 60G15; 60H05

*Keywords:* Gaussian process; Multiple stochastic integral; Product formula; Independence; Fractional Brownian motion

## 1. Introduction

The problem of finding necessary and sufficient conditions for independence of multiple stochastic integrals in terms of their deterministic integrands was initially addressed by Üstünel and Zakai (1989) in the context of the classical multiple Wiener–Itô integrals with respect to Brownian motion. The original question was aimed at characterizing stochastic independence of nonlinear functionals of a Brownian motion, and subsequently inspired a corresponding study of stochastic independence of nonlinear functionals of other stochastic processes. It is well-known that $L^2$-functionals of certain processes (e.g., processes generating independently scattered random measures or possessing a martingale structure) admit chaos expansions, thus can be expressed in the $L^2$ sense as series of multiple stochastic integrals of various orders. Consequently, derivation of independence criteria for multiple stochastic integrals represents a natural step en route to characterizing dependence structure between nonlinear functionals of stochastic processes.

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In the case of multiple Wiener–Itô integrals with respect to a Brownian motion, Üstünel and Zakai (1989) showed that independence of stochastic integrals is equivalent to the $L^2$ orthogonality of one-dimensional sections of corresponding pairs of deterministic symmetric kernels. The result sparked an interest in establishing independence criteria for other types of multiple stochastic integrals. Most notably, Rosiński and Samorodnitsky (1999) considered multiple integrals with respect to an $\alpha$-stable random measure ($0<\alpha<2$) and showed that two such integrals are independent, if and only if the supports of one-dimensional sections of the kernels are disjoint. Another independence criterion was obtained by Privault (1996, 1999) in the case of multiple stochastic integrals with respect to a martingale $M$ satisfying the structure equation $\mathrm{d}[M,M]_t = \mathrm{d}t + h_t\,\mathrm{d}M_t$ for some measurable deterministic function $h$. Namely, multiple integrals $I_n^{\mathrm{M}}(f_n)$ and $I_m^{\mathrm{M}}(g_m)$ of orders $n$ and $m$ of deterministic symmetric kernels $f_n \in L^2(\mathbb{R}_+)$ and $g_m \in L^2(\mathbb{R}_+)$ with respect to $(M_t)_{t\geqslant 0}$ are independent if and only if the following two conditions hold:

(i) $\int_0^\infty f_n(t,x_2,\dots,x_n)g_m(t,x_{n+1},\dots,x_{n+m-2})\,\mathrm{d}t = 0$ a.e. $(\mathrm{d}x_2\dots\mathrm{d}x_{n+m-2})$ and
(ii) $f_n(x_1,x_2,\dots,x_n)g_m(x_1,x_{n+1},\dots,x_{n+m-1}) = 0$ a.e. $(|h_{x_1}|\,\mathrm{d}x_1\,\mathrm{d}x_2\dots\mathrm{d}x_{n+m-1})$.

The aim of the present paper is to establish necessary and sufficient conditions for independence of multiple stochastic integrals with respect to a (general) Gaussian process, where neither orthogonal increments property nor a martingale structure of the integrator is assumed. This class of multiple stochastic integrals was originally introduced by Huang and Cambanis (1978). Note also that, in the special case when the Gaussian integrator is a fractional Brownian motion (fBm), the multiple stochastic (fractional) integrals have also been reintroduced via alternative (e.g., reproducing kernel Hilbert space and Wick calculus) methods and their chaos expansions studied by Dasgupta and Kallianpur (1999a,b) and Duncan et al. (2000) and, more recently, by several other authors.

The paper is organized as follows. Section 2 contains material on the definition and properties of multiple Huang–Cambanis stochastic integrals and a derivation of the product formula for the integrals. Section 3 presents our main result on the independence criterion for multiple Huang–Cambanis integrals defined with respect to a general Gaussian process. Finally, as an illustration, the result is applied to deduce the independence criterion for multiple stochastic integrals with respect to fBm.

## 2. Huang–Cambanis integrals and product formula

Consider a Gaussian process $X = (X_t, t\in\mathbb{T})$ defined on a probability space $(\Omega,\mathscr{F},P)$ with zero mean and covariance function $\gamma(t,s)$, and parametrized by $\mathbb{T}$, which is a finite or infinite interval of the real line. Assume that $\gamma(\cdot,\cdot)$ is a function of bounded variation on bounded domains of $\mathbb{T}^2$ and $X_{t_0}=0$ a.s. for some $t_0\in\mathbb{T}$. We denote by $\otimes$ a tensor product.

Consider the set $\mathscr{E}_n$ of all elementary functions $f$ on $\mathbb{T}^n$ of the form

$$f(t_1,\dots,t_n) = \sum_{i_1,\dots,i_n=1}^{N} f_{i_1,\dots,i_n} 1_{A_{i_1}\times\cdots\times A_{i_n}}(t_1,\dots,t_n),$$

where each $A_{i_k} = (a_{i_k}, b_{i_k}]$ is a bounded interval in $\mathbb{T}$. (Note: the coefficients $f_{i_1,\dots,i_n}$ need not vanish when two of the indices $i_1,\dots,i_n$ are equal.) For all $f,g\in\mathscr{E}_n$ define

$$\langle f,g\rangle_{\gamma,n} = \int_{\mathbb{T}^2}\cdots\int_{\mathbb{T}^2} f(t_1,\dots,t_n)g(s_1,\dots,s_n)\gamma(\mathrm{d}t_1,\mathrm{d}s_1)\dots\gamma(\mathrm{d}t_n,\mathrm{d}s_n),$$

where the integration is carried out in the Riemann–Stieltjes sense. Then $(\mathscr{E}_n,\langle\cdot,\cdot\rangle_{\gamma,n})$ is an inner product space and let us denote by $\Lambda_\gamma^2(\mathbb{T}^n)$ its completion. Also, define $L_\gamma^2(\mathbb{T}^n)$ to be the set of all measurable functions $f$ on $\mathbb{T}^n$ such that the following Lebesgue integrals

$$\int_{\mathbb{T}^{2n}} |f(t_1,\dots,t_n)f(s_1,\dots,s_n)|\,|\gamma|(\mathrm{d}t_1,\mathrm{d}s_1)\dots|\gamma|(\mathrm{d}t_n,\mathrm{d}s_n)<\infty$$

and

$$\int_{\mathbb{T}^{2n}} |f(t_1,\dots,t_n)| 1_{A_1\times\cdots\times A_n}(s_1,\dots,s_n)|\gamma|(\mathrm{d}t_1,\mathrm{d}s_1)\dots|\gamma|(\mathrm{d}t_n,\mathrm{d}s_n)<\infty$$

are finite for all $A_1\times\cdots\times A_n\in\mathbb{T}^n$, with each $A_i\subset\mathbb{T}$ being a bounded half open interval. Then it follows from Huang and Cambanis (1978) that $L^2_\gamma(\mathbb{T}^n)$ is dense in $\Lambda^2_\gamma(\mathbb{T}^n)$, and $\forall f,g\in L^2_\gamma(\mathbb{T}^n)$ such that $\int_{\mathbb{T}^{2n}}|f(t_1,\dots,t_n)\, g(s_1,\dots,s_n)|\,|\gamma|(\mathrm{d}t_1,\mathrm{d}s_1)\dots|\gamma|(\mathrm{d}t_n,\mathrm{d}s_n)<\infty$, their scalar product is equal to

$$\langle f,g\rangle_{\gamma,n}=\int_{\mathbb{T}^{2n}} f(t_1,\dots,t_n)g(s_1,\dots,s_n)\gamma(\mathrm{d}t_1,\mathrm{d}s_1)\dots\gamma(\mathrm{d}t_n,\mathrm{d}s_n),$$

where the integration is in the Lebesgue sense.

For any given function $f\in\Lambda^2_\gamma(\mathbb{T}^n)$, let us define a symmetrization of $f$ by $\widetilde{f}(t_1,\dots,t_n)=(n!)^{-1}\sum_\pi f(t_{\pi_1},\dots,t_{\pi_n})$, where the sum is taken over all permutations $\pi=(\pi_1,\dots,\pi_n)$ of $\{1,\dots,n\}$. Note that a typical element in $\Lambda^2_\gamma(\mathbb{T}^n)$ is a limit point of a Cauchy sequence of functions from $\mathscr{E}_n$ and may fail to be a function. However, due to the inequality $\|\widetilde{f}\|^2_{\gamma,n}\leqslant n!\|f\|^2_{\gamma,n}$, the definition of symmetrization can be extended by continuity from $\mathscr{E}_n$ to $\Lambda^2_\gamma(\mathbb{T}^n)$. Denote by $\Lambda^2_{\gamma,s}(\mathbb{T}^n)$ the class of elements $f\in\Lambda^2_\gamma(\mathbb{T}^n)$ such that $f=\widetilde{f}$. Similarly, let $L^2_{\gamma,s}(\mathbb{T}^n)$ consist of all functions $f\in L^2_\gamma(\mathbb{T}^n)$ that are symmetric.

For an arbitrary $f\in\mathscr{E}_1$, $f(t)=\sum_1^N f_n 1_{(a_n,b_n]}(t)$, where $(a_n,b_n]\subset\mathbb{T}$, define

$$I_1^X(f):=\int_{\mathbb{T}} f(t)\,\mathrm{d}X_t=\sum_1^N f_n(X_{b_n}-X_{a_n}).$$

Then $\mathbb{E}I_1^X(f)=0$ and $E(I_1^X(f)I_1^X(g))=\int_{\mathbb{T}^2}f(t)g(s)\gamma(\mathrm{d}t,\mathrm{d}s)=\langle f,g\rangle_{\gamma,1}$. Then $I_1^X$ can be extended to an isometric isomorphism (also denoted by $I_1^X$) between the Hilbert space $\Lambda^2_\gamma(\mathbb{T}^1)$ and the linear space of $X$, $\mathscr{H}(X)$, which is the closed subspace of $L^2(X)$ spanned by $X_t, t\in\mathbb{T}$. Next let $\{e_i\}_{i=1}^\infty$ be a complete orthonormal system in the pre-Hilbert separable space $L^2_\gamma(\mathbb{T}^1)$ and let $f(t_1,\dots,t_n)=\sum_{1\leqslant k_i\leqslant k}a_{k_1,\dots,k_n}e_{k_1}(t_1)\dots e_{k_n}(t_n)$ be symmetric. Then $f\in L^2_{\gamma,s}(\mathbb{T}^n)$ and

$$I_n^X(f):=\sum_{1\leqslant k_i\leqslant k}a_{k_1,\dots,k_n}\left(I_1^X\right)^{\diamond 1}(e_{k_1})\diamond\cdots\diamond\left(I_1^X\right)^{\diamond 1}(e_{k_n})$$

defines the $n$th order Huang–Cambanis integral of $f$ with respect to $(X_t, t\in\mathbb{T})$. (Here '$\diamond$' denotes the Wick product and Wick powers in the sense of Simon (1974).) By denseness of $L^2_{\gamma,s}(\mathbb{T}^n)$ in $\Lambda^2_{\gamma,s}(\mathbb{T}^n)$ and $E(I_n^X(f)^2)=n!\|f\|^2_{\gamma,n}$ for $f\in L^2_{\gamma,s}(\mathbb{T}^n)$, the operator $I_n^X$ can then be extended to a linear and continuous operator from $\Lambda^2_{\gamma,s}(\mathbb{T}^n)$ to $L^2(\Omega,\mathscr{F},P)$. Moreover, for arbitrary nonsymmetric $f\in\Lambda^2_\gamma(\mathbb{T}^n)$, define its Huang–Cambanis multiple stochastic integral with respect to $X$ by $I_n^X(f):=I_n^X(\tilde{f})$. The following properties of the multiple Huang–Cambanis integrals (shown in Huang and Cambanis, 1978) are important here:

(HC$_1$): For arbitrary $f,g\in\Lambda^2_\gamma(\mathbb{T}^n)$, $I_n^X(af+bg)=aI_n^X(f)+bI_n^X(g)\quad\forall a,b\in\mathbb{R}$.
(HC$_2$): For all $f\in\Lambda^2_\gamma(\mathbb{T}^n)$, $g\in\Lambda^2_\gamma(\mathbb{T}^m)$, the integrals satisfy the equalities $E(I_n^X(f))=0$ and

$$E(I_n^X(f)I_m^X(g))=\begin{cases} n!\langle\widetilde{f},\widetilde{g}\rangle_{\gamma,n} & \text{if } m=n,\\ 0 & \text{if } m\neq n.\end{cases}$$

(HC$_3$): For arbitrary $h\in\Lambda^2_\gamma(\mathbb{T}^1)$, $I_n^X(h^{\otimes n})=\|h\|^n_{\gamma,1}H_n\left(I_1^X(h)/\|h\|_{\gamma,1}\right)$, where $H_n$ is the $n$th Hermite polynomial.

(HC$_4$): $I_n^X(h_1^{\otimes n_1}\otimes\cdots\otimes h_k^{\otimes n_k})=\Pi_{j=1}^k H_{n_j}(I_1^X(h_j))$ for all orthonormal $h_1,\dots,h_k\in\Lambda^2_\gamma(\mathbb{T}^1)$ (i.e. $\langle h_i,h_j\rangle_{\gamma,1}=0$ for all $i\neq j$ and $\|h_j\|_{\gamma,1}=1$ for all $j$), where $n=n_1+\cdots+n_k$.

**Theorem 1** (*Product formula*). *For all* $f \in L^2_{\gamma,s}(\mathbb{T}^n)$, $g \in L^2_{\gamma,s}(\mathbb{T}^m)$,

$$I_n^X(f)I_m^X(g) = \sum_{k=0}^{m\wedge n} k!\binom{n}{k}\binom{m}{k} I^X_{n+m-2k}(f\otimes_{\gamma,k}g),$$

*where* $\otimes_{\gamma,k}$ *denotes the kth order* $\gamma$*-contraction defined by*:

$$f\otimes_{\gamma,k}g(t_1,\dots,t_{n+m-2k}) := \int_{\mathbb{T}^{2k}} f(t_1,\dots,t_{n-k},s_1,\dots,s_k) \times g(s'_1,\dots,s'_k,t_{n-k+1},\dots,t_{n+m-2k})\gamma(\mathrm{d}s_1,\mathrm{d}s'_1)\dots\gamma(\mathrm{d}s_k,\mathrm{d}s'_k)$$

*for all* $1 \leqslant k \leqslant (m\wedge n)$, $f\otimes_{\gamma,0}g := f\otimes g$, *and* $I_0^X$ *denotes the identity operator.*

**Proof.** The proof is analogous to the proof of the product formula for the multiple Wiener–Itô integrals in Shigekawa (1980). We will outline the key steps here. First, note that $\forall f,g \in L^2_\gamma(\mathbb{T})$,

$$\begin{aligned}
I_2^X(f\otimes g) &= \tfrac{1}{2}I_2^X((f+g)^{\otimes 2} - f^{\otimes 2} - g^{\otimes 2})\\
&= \frac{1}{2}\left[\|f+g\|^2_{\gamma,1}H_2\left(\frac{I_1^X(f+g)}{\|f+g\|_{\gamma,1}}\right) - \|f\|^2_{\gamma,1}H_2\left(\frac{I_1^X(f)}{\|f\|_{\gamma,1}}\right) - \|g\|^2_{\gamma,1}H_2\left(\frac{I_1^X(g)}{\|g\|_{\gamma,1}}\right)\right]\\
&= \tfrac{1}{2}[(I_1^X(f+g))^2 - \|f+g\|^2_{\gamma,1} - (I_1^X(f))^2 + \|f\|^2_{\gamma,1} - (I_1^X(g))^2 + \|g\|^2_{\gamma,1}]\\
&= I_1^X(f)I_1^X(g) - \langle f,g\rangle_{\gamma,1}.
\end{aligned}$$

Then, by $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$ and induction, it follows that $\forall n \geqslant 1$,

$$I_n^X(f^{\otimes n})I_1^X(g) = I^X_{n+1}(f^{\otimes n}\otimes g) + n\langle f,g\rangle_{\gamma,1}\, I^X_{n-1}(f^{\otimes n-1}).$$

Further, by induction (on $m$), it is easy to check that for all $n,m \geqslant 1$,

$$I_n^X(f^{\otimes n})I_m^X(f^{\otimes m}) = \sum_{k=0}^{n\wedge m} k!\binom{n}{k}\binom{m}{k}\langle f,f\rangle^k_{\gamma,1}\, I_{n+m-2k}(f^{\otimes n+m-2k}).$$

Finally, after some elementary algebra and use of property ($HC_4$), it follows that for all $f,g \in L^2_\gamma(\mathbb{T}^1)$,

$$\begin{aligned}
I_n^X(f^{\otimes n})I_m^X(g^{\otimes m}) &= \sum_{k=0}^{m\wedge n} k!\binom{n}{k}\binom{m}{k}\langle f,g\rangle^k_{\gamma,1} I^X_{n+m-2k}(f^{\otimes n-k}\otimes g^{\otimes m-k})\\
&= \sum_{k=0}^{m\wedge n} k!\binom{n}{k}\binom{m}{k} I^X_{n+m-2k}(f^{\otimes n}\otimes_{\gamma,k} g^{\otimes m}).
\end{aligned}$$

Therefore, the product formula holds for any two functions in $L^2_{\gamma,s}(\mathbb{T}^n)$ that are powers (in the sense of $\otimes$) of functions from $L^2_\gamma(\mathbb{T}^1)$. Linear combinations of such functions are dense in $L^2_{\gamma,s}(\mathbb{T}^n)$, for all $n \geqslant 1$, thus the result holds for any two functions from $L^2_{\gamma,s}(\mathbb{T}^n)$ and $L^2_{\gamma,s}(\mathbb{T}^m)$. □

In order to extend the product formula to the case of integrands $f \in \Lambda^2_{\gamma,s}(\mathbb{T}^n)$ and $g \in \Lambda^2_{\gamma,s}(\mathbb{T}^m)$, which may no longer be functions, one needs to extend the notion of the $k$th order $\gamma$-contraction. First note that $\forall f \in L^2_\gamma(\mathbb{T}^n)$,

$$\begin{aligned}
\|f\otimes_{\gamma,k}f\|^2_{\gamma,2(n-k)} &= \int_{\mathbb{T}^{4(n-k)}}\left[\int_{\mathbb{T}^{4k}} f(x_1,\dots,x_{n-k},u_1,\dots,u_k)f(x'_1,\dots,x'_{n-k},u'_1,\dots,u'_k)\right.\\
&\quad\times f(y_1,\dots,y_{n-k},v_1,\dots,v_k)f(y'_1,\dots,y'_{n-k},v'_1,\dots,v'_k)\gamma(\mathrm{d}u_1,\mathrm{d}u'_1)\dots\gamma(\mathrm{d}u_k,\mathrm{d}u'_k)\\
&\quad\left.\times\gamma(\mathrm{d}v_1,\mathrm{d}v'_1)\dots\gamma(\mathrm{d}v_k,\mathrm{d}v'_k)\right]\gamma(\mathrm{d}x_1,\mathrm{d}y_1)\dots\gamma(\mathrm{d}x_{n-k},\mathrm{d}y_{n-k})\gamma(\mathrm{d}x'_1,\mathrm{d}y'_1)\dots\gamma(\mathrm{d}x'_{n-k},\mathrm{d}y'_{n-k})\\
&= \langle q_f,\widehat{q}_f\rangle_{\gamma,2n} \leqslant \|q_f\|_{\gamma,2n}\|\widehat{q}_f\|_{\gamma,2n} = \|f\|^2_{\gamma,n},
\end{aligned}$$

where $q_f(t_1,\ldots,t_{2n})=f(t_1,\ldots,t_n)f(t_{n+1},\ldots,t_{2n})$ and

$$\widehat{q}_f(t_1,\ldots,t_{2n})=f(t_1,\ldots,t_{n-k},t_{2n-k+1},\ldots,t_{2n})f(t_{n+1},\ldots,t_{2n-k},t_{n-k+1},\ldots,t_n).$$

Similarly one shows that $\forall f\in L^2_\gamma(\mathbb{T}^n)$, $\forall g\in L^2_\gamma(\mathbb{T}^m)$, the inequality

$$\|f\otimes_{\gamma,k}g\|^2_{\gamma,n+m-2k}\leqslant\|f\|_{\gamma,n}\|g\|_{\gamma,m}$$

holds for each $0\leqslant k\leqslant(m\wedge n)$. Therefore, denseness of $L^2_\gamma(\mathbb{T}^n)$ in $\Lambda^2_\gamma(\mathbb{T}^n)$ for all $n$ allows to extend the definition of $\otimes_{\gamma,k}$ for arbitrary elements $f\in\Lambda^2_\gamma(\mathbb{T}^n)$, $g\in\Lambda^2_\gamma(\mathbb{T}^m)$ by continuity. Therefore, the following is an immediate consequence of Theorem 1.

**Corollary 1.** *For arbitrary $f\in\Lambda^2_{\gamma,s}(\mathbb{T}^n)$, $g\in\Lambda^2_{\gamma,s}(\mathbb{T}^m)$,*

$$I^X_n(f)I^X_m(g)=\sum_{k=0}^{m\wedge n}k!\binom{n}{k}\binom{m}{k}I^X_{n+m-2k}(f\otimes_{\gamma,k}g).$$

## 3. Independence criterion for Huang–Cambanis integrals

**Theorem 2.** *Consider arbitrary $f\in L^2_{\gamma,s}(\mathbb{T}^n)$, $g\in L^2_{\gamma,s}(\mathbb{T}^m)$, $n,m\geqslant 1$. Then the following statements are equivalent*:

(i) *$I^X_n(f)$ and $I^X_m(g)$ are independent*;
(ii) *$f\otimes_{\gamma,1}g=0$ a.e. on $\mathbb{T}^{n+m-2}$*;
(iii) $\mathrm{Cov}([I^X_n(f)]^2,[I^X_m(g)]^2)=0$.

**Proof.** Clearly, (i) implies (iii). To see that (iii) implies (ii), note that $\forall f\in L^2_{\gamma,s}(\mathbb{T}^n)$, $g\in L^2_{\gamma,s}(\mathbb{T}^m)$, as in Lemma 4.2 in Rosiński and Samorodnitsky (1999),

$$\|\widetilde{f\otimes g}\|^2_{\gamma,n+m}=\sum_{k=0}^{n\wedge m}\left[\binom{n}{k}\binom{m}{k}\Big/\binom{n+m}{n}\right]\|f\otimes_{\gamma,k}g\|^2_{\gamma,n+m-2k}.$$

Taking into account property ($\mathrm{HC}_2$) and the product formula, it follows that

$$E[(I^X_n(f)I^X_m(g))^2]\geqslant n!m!\|f\otimes g\|^2_{\gamma,n+m}+n!m!nm\|f\otimes_{\gamma,1}g\|^2_{\gamma,n+m-2}.$$

Therefore,

$$\mathrm{Cov}((I^X_n(f))^2,(I^X_m(g))^2)\geqslant n!m!nm\|f\otimes_{\gamma,1}g\|^2_{\gamma,n+m-2},$$

thus, (iii) $\Rightarrow$ (ii). Now let us show that (ii) implies (i). Let $H_f$ be the pre-Hilbert space in $L^2_\gamma(\mathbb{T}^1)$ spanned by functions

$$s\mapsto\int_{\mathbb{T}^{2(n-1)}}f(x_1,\ldots,x_{n-1},s)h(y_1,\ldots,y_{n-1})\gamma(\mathrm{d}x_1,\mathrm{d}y_1)\ldots\gamma(\mathrm{d}x_{n-1},\mathrm{d}y_{n-1})$$

for $h\in L^2_\gamma(\mathbb{T}^{n-1})$. Define similarly $H_g$. Then it is easy to see that if $f\otimes_{\gamma,1}g=0$ a.e. then $H_f$ and $H_g$ are orthogonal. Moreover, $f$ (respectively, $g$) can be expressed as a sum of tensor products of elements in $H_f$ (respectively, $H_g$). Pick an orthonormal basis in $H_f$ and an orthonormal basis in $H_g$. Then, by property ($\mathrm{HC}_4$), $I^X_n(f)$ (respectively, $I^X_m(g)$) can be written as a sum of products of Hermite polynomials (multiplied by suitable constants) of the first-order integrals of the basis elements in $H_f$ (respectively, $H_g$). In view of the isometry property of $I^X_1$ and orthogonality of $H_f$ and $H_g$, the first-order integrals of elements of $H_f$ must be orthogonal to the first-order integrals of elements of $H_g$. Since the first-order integrals are Gaussian variables, their orthogonality implies independence, which in turn implies independence of $I^X_n(f)$ and $I^X_m(g)$. $\square$

**Corollary 2.** *Let $(X_t,t\in\mathbb{T})$ be a fractional Brownian motion $(B^H_t,t\in[0,T])$ with Hurst index $H\in(0,1)$ such that $H\neq\frac{1}{2}$. Then, $\gamma(s,t)=\frac{1}{2}(s^{2H}+t^{2H}-|t-s|^{2H})$ and $(\partial^2/\partial t\partial s)\gamma(t,s)=\phi(t,s)$ a.e., where*

$$\phi(t,s)=H(2H-1)|t-s|^{2H-2}.$$

*Thus, in the persistent case, i.e. when* $H > \frac{1}{2}$, $\gamma$ *is a function of bounded variation on* $[0, T]^2$ *and* $\gamma(\mathrm{d}t, \mathrm{d}s) = \phi(t, s)\,\mathrm{d}t\,\mathrm{d}s$. *Therefore,* $I_n^{B^H}(f)$ *and* $I_m^{B^H}(g)$ *are independent, if and only if*

$$\int_{[0,T]^2} f(t_1, \ldots, t_{n-1}, s) g(s', t_n, \ldots, t_{n+m-2}) |s - s'|^{2H-2}\,\mathrm{d}s\,\mathrm{d}s' = 0 \quad \text{a.e.}$$

Note that in the anti-persistent case, i.e. $H < \frac{1}{2}$, $\phi$ is not integrable over $[0, T]^2$ and $\gamma$ is no longer a function of bounded variation on $[0, T]^2$.

**Remark.** The independence criterion for multiple integrals with respect to a persistent fBm $(B_t^H, t \in [0, T])$, obtained in the above Corollary, could also be deduced from the independence criterion, available for the classical multiple Wiener–Itô integrals, with the help of the stochastic transfer principle established by Perez-Abreu and Tudor (2002). Note, however, that no stochastic transfer principle currently exists in the case of multiple stochastic integration with respect to a general Gaussian process.

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