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LARGE DEVIATION PRINCIPLE FOR A STOCHASTIC NAVIER-STOKES EQUATION IN ITS VORTICITY FORM FOR A TWO-DIMENSIONAL INCOMPRESSIBLE FLOW

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Abstract. We derive a large deviation principle for a stochastic Navier-Stokes equation for the vorticity of a two-dimensional fluid when the magnitude of the random term tends to zero. The key is the verification of the exponential tightness for the stochastic vorticity.

1. **Introduction.** Central position in the study of classical fluid dynamics is occupied by the Navier-Stokes equations, which are a system of nonlinear partial differential equations thought to comprise a fundamental dynamical description of viscous fluid motion for both laminar and turbulent flows. Turbulence is associated with high velocities and highly irregular or chaotic fluid paths observed at multiple scales, apparently caused by high sensitivity to initial conditions and to perturbations, by occurrence of instabilities, and an interplay of large numbers of degrees of freedom and scales. Experimental results show that for large Reynolds numbers existing microscopic perturbations get amplified to macroscopic scales giving rise to unsteady flows whose observed behavior deviates significantly from predictions made on the basis of the classical Navier-Stokes model. As a result, in an attempt to design evolution equations that would better describe and predict turbulent behavior displayed by fluids, the study of stochastic Navier-Stokes models was initiated in the 1970s.

Randomly forced Navier-Stokes models. First stochastic Navier-Stokes model was suggested by Bensoussan and Temam in 1973 in [3]. The model was obtained by adding a random force of white noise type to the classical equation. Since then the problem has received wide attention (see e.g. Capinski and Gatarek [4], Flandoli

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and Tortorelli [9], Mikulevicius and Rozovskii [17]) and has been further generalized. Namely for a viscous homogeneous incompressible fluid in a bounded open domain $\mathcal{D} \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\mathcal{D}$, the motion of the fluid in the time interval $[0, T]$ is assumed to be governed by

$$\begin{cases} \frac{\partial u}{\partial t} = \nu\Delta u - (u \cdot \nabla)u - \frac{1}{\rho}\nabla p + f + c(u)\frac{dW(t)}{dt}, \\ \operatorname{div}(u) = 0 \end{cases}$$

for $t \in (0, T)$, supplemented with suitable boundary and initial conditions. In the above model there are two forcing terms: a deterministic component f and a stochastic one of intensity $c(u)$. $W(t)$ stands for a standard one-dimensional Wiener process, u for velocity, p for pressure, ν is the kinematic viscosity of the fluid and ρ is the fluid's (constant) density. Often the white noise term is replaced with a space-time white noise. Mathematically, above SPDE has to be understood in a generalized sense as an evolution equation in certain Sobolev spaces.

A less known approach to "stochastisizing" the Navier-Stokes system is suggested by vortex theory.

Classical vorticity equation. It is a well-known fact that the classical incompressible Navier-Stokes model:

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = \nu\Delta u, \\ \operatorname{div}(u) = 0, \end{cases} \tag{1}$$

in the domain $\mathcal{D} = \mathbb{R}^d$ $(d = 2, 3)$ has an *equivalent* vorticity formulation. Specifically, let $\omega(t, x) = \operatorname{curl}(u(t, x))$, then $\omega(t, x)$ is called a vorticity field and describes how fluid particles are rotating. In particular, when $d = 2$ and the Navier-Stokes system is supplemented with the boundary condition $u(t, x) \to 0$ as $|x| \to \infty$ for all t, the Navier-Stokes system for velocity u can be rewritten in terms of ω as

$$\begin{cases} \frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = \nu\Delta\omega, \\ u(t, x) = \int K(x - y)\omega(t, y)dy, \end{cases} \tag{2}$$

where $K(\cdot)$ is the Biot-Savart kernel given by $K(r) = \nabla^\perp g(|r|)$ with $\nabla^\perp \equiv \binom{\partial_2}{-\partial_1}$ and $g(|r|) = -\frac{1}{2\pi}\ln|r|$ (g is the Green function for a Laplacian in $\mathcal{D}$), $\forall r \in \mathbb{R}^2$. The nonlinear parabolic PDE (2) satisfied by scalar $\omega(t, x) = \partial_1 u_2(t, x) - \partial_2 u_1(t, x)$ (scalar since all other components of $\operatorname{curl}(u)$ vanish in $\mathbb{R}^2$ due to the incompressibility condition) is called a vorticity equation or a Navier-Stokes equation in its vorticity form in $\mathbb{R}^2$.

In this paper we focus our attention on a suitable stochastic version of the vorticity equation (2), we consider its asymptotic behavior as $\nu \to 0$ (inviscid limit) and derive the large deviation principle (LDP) for it.

Point vortex theory approach to Euler's equation. It is often natural to study situations in which vorticity profiles are sharply concentrated around certain points called point vortices. Consider a system of N interacting vortices positioned at $x_i(t) = (x_i^1(t), x_i^2(t))'$ at time t, $i = 1, \ldots, N$. Each vortex has an intensity or "charge" $a_i \in \mathbb{R}$ associated to it (its magnitude is determined by the amount of circulation around the vortex, its sign determines the direction of rotation). The system of point vortices generates an empirical vorticity signed measure-valued

process

$$\omega_t(dx) = \sum_{i=1}^{N} a_i \delta_{x_i(t)}(dx), \tag{3}$$

where $\delta_c(\cdot)$ denotes the Dirac point measure of mass 1 concentrated at $c \in \mathbb{R}^2$. Let each vortex follow the fluid paths with velocity u, that is

$$\begin{cases} \frac{d}{dt} x_i(t) = u(t, x_i(t)), \\ u(t,x) = \int K(x-y)\omega_t(dy), \end{cases} \tag{4}$$

where we also put $K(0) = 0$ (resulting in a slightly modified Biot-Savart law in order to avoid singularity of $K(\cdot)$ at 0). Then it is easy to see that for all $f \in C_b^1(\mathbb{R}^2)$ and $\forall t > 0$,

$$d\langle \omega, f\rangle = \langle \omega, u \cdot \nabla f\rangle dt,$$

which is essentially a weak form of the Euler's equation in its vorticity form, i.e. a weak form of (2) for $\nu = 0$. The latter observation served as a point of origin for the development of modern vortex methods (both deterministic and stochastic) which are primary tools for numerical simulations of inviscid and viscous flows. It also suggested a natural alternative to the randomly forced Navier-Stokes systems discussed earlier.

Stochastic vorticity model in $\mathbb{R}^2$. Let us replace the deterministic system (4), describing the evolution of point vortices, with a stochastic one. The system will naturally generate (via (3)) a stochastic vorticity field ω, whose evolution is described by a stochastic partial differential equation (SPDE) called a stochastic vorticity equation, or a stochastic Navier-Stokes equation (SNSE) in its vorticity form. The very first vorticity SPDE was suggested by Kotelenez in [14] in 1995. Later the model was generalized by Amirdjanova [1] (also Amirdjanova and Kallianpur [2]) to allow for possible jump-discontinuities in the vorticity evolution, and a suitable diffusion approximation to it was suggested.

The latter diffusion is the object of our LDP study in the present paper. Namely, let X^ε represent a stochastic vorticity for a two-dimensional incompressible homogeneous flow with small viscosity $\nu = \epsilon/2$. The process X^ε is a signed measure valued process governed by the SNSE of the form:

$$\begin{cases} d\langle X_t^\varepsilon, f\rangle &= \left\langle X_t^\varepsilon, \frac{\varepsilon}{2}\Delta f + (U_\delta X_t^\varepsilon)\cdot\nabla f\right\rangle dt \\ & \quad + \left\langle X_t^\varepsilon, \frac{\varepsilon}{2}\sum_{k,\ell=1}^2 (H(t,\cdot)H^T(t,\cdot))_{k\ell}\partial_{k\ell}^2 f\right\rangle dt \\ & \quad + \sqrt{\varepsilon}\sum_{j=1}^2 \int_{\mathbb{R}^2} \langle X_t^\varepsilon, \Gamma(\cdot,v)\partial_j f\rangle W^j(dvdt) \\ & \quad + \sqrt{\varepsilon}\langle X_t^\varepsilon, H(t,\cdot)\cdot\nabla f\rangle dW^0(t), \\ X_0^\varepsilon &= x_0 \in L^2(\mathbb{R}^2)\cap\mathcal{M}(a) \end{cases} \tag{5}$$

for all $f \in C_b^2(\mathbb{R}^2)$, where W^1 and W^2 are independent, space-time Brownian motions, independent of the 1-dimensional Brownian motion W^0,

$$U_\delta\mu(r) = \int_{\mathbb{R}^2} K_\delta(r-q)\mu(dq), \qquad K_\delta(r) = \begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix} g_\delta(|r|)$$

and $\langle \mu, f\rangle$ denotes $\int f d\mu$. The space $L^2(\mathbb{R}^2)\cap\mathcal{M}(a)$ represents the space of all finite positive or negative signed measures μ defined on Borel subsets of $\mathbb{R}^2$, such that μ is absolutely continuous with respect to the Lebesgue measure λ with the Radon-Nikodym derivative $d\mu/d\lambda \in L^2(\mathbb{R}^2)$ and $\mu(\mathbb{R}^2) = a$. (Fixed real constant a indicates the total amount and direction of rotation (circulation) of the fluid in $\mathbb{R}^2$.

Note that, in the above definition, $\mu(B) \geq 0$ for all Borel subsets B of $\mathbb{R}^2$ if $a > 0$, and $\mu(B) \leq 0$ for all Borel subsets B of $\mathbb{R}^2$ if $a < 0$.) The initial vorticity profile x_0 is assumed to be deterministic.

We make the following assumptions on the coefficients:
(A1). g_δ is at least a twice continuously differentiable approximation to function $g(|r|) = -\frac{1}{2\pi}\log|r|$, with bounded derivatives up to order 2, satisfying conditions $|g'_\delta(\tau)| \leq |g'(\tau)|$ and $|g''_\delta(\tau)| \leq |g''(\tau)|$, for all $\tau > 0$; $g_\delta(\tau) = g(\tau)$ for all $\tau \geq \delta$.
(A2). $\Gamma:\ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a bounded measurable function, $\Gamma(r,v) = \Gamma(v,r)$ and

$$\int_{\mathbb{R}^2} \Gamma(v,r)^2 dv = 1, \qquad \int_{\mathbb{R}^2} |\Gamma(v,r) - \Gamma(q,r)|^2 dr \leq C|v-q|^2.$$

(A3). $H:\ \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is a measurable function such that

$$|H(t,v)| \leq C \text{ and } |H(t,v) - H(t,q)| \leq C|v-q|.$$

It follows from (5) that when $\varepsilon \to 0$, X_t^ε converges in probability to ω_t given by

$$d\langle \omega_t, f\rangle = \langle \omega_t, (U_\delta \omega_t)\cdot \nabla f\rangle\, dt$$

which is deterministic (in fact, the limit is a solution to a δ-regularized weak Euler equation). In this paper we study the large deviation principle (LDP) for $\{X^\varepsilon\}$ on $C([0,T],\mathcal{M}(a))$. From the point of view of applications, it is of interest to study the convergence rate. One of the methods is the LDP, i.e., to prove the above convergence at an exponential rate and to identify the rate function.

The LDP for various important examples of measure-valued processes has been studied by several authors. Fleischmann et al [10], Fleischmann and Kaj [11] and Schied [18] studied the LDP for super-Brownian motions. Dawson and Feng [5] and Feng and Xiong [8] considered the LDP for Fleming-Viot processes. Dawson and Gärtner [6] investigated the LDP for the empirical measure process of a particle system with mean-field interaction. Xiong [20] derived a LDP in the context of optimal nonlinear filtering. In the present setup, the processes take values in the space $\mathcal{M}(a)$ of signed measures with a fixed total mass a. (This reflects an important physical property of an incompressible motion in $\mathbb{R}^2$, namely, conservation of total vorticity.)

This article is organized as follows: In Section 2, we give some preliminary material and state our main result. In Section 3, we present a detailed treatment of the SNSE. In Sections 4 and 5, we verify two crucial conditions, namely the exponential continuity of a map and the exponential tightness of the processes, in deriving LDP. Finally, in Section 6, we complete the proof of the main theorem.

2. **LDP framework and main results.**

First let us establish a general framework needed for the LDP result. Let $\mathcal{X} \subset \mathcal{Y}$ be metric spaces with metrics $d_{\mathcal{X}}$ and $d_{\mathcal{Y}}$, and $\mathcal{Y}$ is a linear space. Let $\mathcal{Z}$ be a Banach space. Let $\mathcal{A}_{\mathcal{X}}$, $\mathcal{A}_{\mathcal{Y}}$ and $\mathcal{A}_{\mathcal{Z}}$ be sets of $\mathcal{X}$-, $\mathcal{Y}$- and $\mathcal{Z}$-valued random variables. Let $X^\varepsilon \in \mathcal{A}_{\mathcal{X}}$ and $Z^\varepsilon \in \mathcal{A}_{\mathcal{Z}}$ be such that

$$X^\varepsilon = F(X^\varepsilon, Z^\varepsilon)\ \ \forall \varepsilon, \tag{6}$$

where $F:\ \mathcal{A}_{\mathcal{X}} \times \mathcal{A}_{\mathcal{Z}} \to \mathcal{A}_{\mathcal{Y}}$. We recall the definition of the LDP briefly and refer the reader to [7] for details.

Definition 1. *$\{X^\varepsilon\}$ satisfies the LDP with rate function $I:\ \mathcal{X} \to [0,\infty]$ if:*
i) For any $c > 0$, the set $\{x \in \mathcal{X}:\ I(x) \leq c\}$ is a compact subset of $\mathcal{X}$.
ii) For any open subset G of $\mathcal{X}$, we have

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq -\inf_G I(x).$$

iii) For any closed subset C of $\mathcal{X}$, we have

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^\varepsilon\in C)\le -\inf_C I(x).$$

We make the following Assumptions (L):
(L1) (Exponential tightness) $\forall L>0$, $\exists$ a compact subset C_L of $\mathcal{X}$ such that

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^\varepsilon\notin C_L)\le -L.$$

(L2) (Exponential continuity) $\forall L>0$, $\forall\delta>0$, $\exists\delta'>0$ s.t. $\forall X_1,\ X_2\in\mathcal{A}_{\mathcal{X}}$,

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(d_{\mathcal{X}}(X_1,X_2)<\delta',\ d_{\mathcal{Y}}(F(X_1,Z^\varepsilon),F(X_2,Z^\varepsilon))>\delta)\le -L.$$

(L3) (Absolute continuity) Let $\mathcal{H}\subset\mathcal{Z}$ be a Hilbert space. $\forall\ell\in\mathcal{H}$, we have

$$\frac{dQ_\ell}{dQ}=\exp\left(\xi^\varepsilon_\ell-\frac{1}{2\varepsilon}\|\ell\|^2_{\mathcal{H}}\right),$$

where $Q_\ell=P\circ(Z^\varepsilon+\ell)^{-1}$, $Q=P\circ(Z^\varepsilon)^{-1}$, ξ^ε_ℓ is a real-valued random variable such that $\mathbb{E}^Q(\xi^\varepsilon_\ell)=0$ and $\mathbb{E}^Q(\xi^\varepsilon_\ell)^2=\frac{1}{\varepsilon}\|\ell\|^2_{\mathcal{H}}$.
(L4) (Uniqueness) For each $\ell\in\mathcal{H}$, the following equation

$$x=F(x,\ell),$$

has a unique solution in $\mathcal{X}$, denoted by $x=\gamma(\ell)$.
(L5) (Convergence) Let Y^ε be $\mathcal{X}$-valued random variables satisfying

$$Y^\varepsilon=F(Y^\varepsilon,Z^\varepsilon+\ell).$$

Then $Y^\varepsilon\to\gamma(\ell)$ in probability, as $\varepsilon\to 0$.
(L6) (LDP for frozen processes) $\forall x\in\mathcal{X}$, $X^{x,\varepsilon}\equiv F(x,Z^\varepsilon)\in\mathcal{A}_{\mathcal{Y}}$ satisfies the LDP with rate function

$$I^x(y)=\inf\left\{\frac{1}{2}\|\ell\|^2_{\mathcal{H}}:\ y=F(x,\ell)\right\}.$$

Recall the following result proved in ([19]; See also [20]).

Proposition 1. *Under Assumptions (L), $\{X^\varepsilon\}$ in (6) satisfies the LDP with rate function*

$$I(x)=\inf\left\{\frac{1}{2}\|\ell\|^2_{\mathcal{H}}:\ x=\gamma(\ell)\right\}.$$

Next let $\mathcal{M}(a)$ be a collection of all finite positive or negative signed measures μ on Borel subsets of $\mathbb{R}^2$ such that $\mu(\mathbb{R}^2)=a$, where $a\in\mathbb{R}$ is a fixed constant. For any finite signed measures $\mu_1,\ \mu_2$, the Fortet-Mourier (also called Monge-Kantorovich-Rubinstein or Wasserstein) metric is defined by

$$\rho_W(\mu_1,\mu_2)=\sup\{|\langle\mu_1,f\rangle-\langle\mu_2,f\rangle|:\ f\in\mathbb{B}_1\}$$

where

$$\mathbb{B}_1=\{f:\ |f(v)-f(q)|\le|v-q|,\ \ |f(v)|\le 1\ \forall v,q\in\mathbb{R}^2\}.$$

Then $\mathcal{M}(a)$ is a convex set and $(\mathcal{M}(a),\rho_W)$ is a complete separable metric space with the topology equivalent to the usual weak convergence topology. Also let $C^2_0(\mathbb{R}^2):=\left\{f\in C^2_b(\mathbb{R}^2):\lim_{|v|\to\infty}f(v)=\lim_{|v|\to\infty}\partial_i f(v)=\lim_{|v|\to\infty}\partial^2_{ij}f(v)=0\ \ \forall i,j=1,2\right\}$.

Then $C_0^2(\mathbb{R}^2)$ is a closed linear subspace of $C_b^2(\mathbb{R}^2)$, which (unlike $C_b^2(\mathbb{R}^2)$) is separable under the natural sup norm in $C_b^2(\mathbb{R}^2)$. Let $\{f_j\}_{j=1}^{\infty} \subset C_0^2(\mathbb{R}^2)$ be a family whose spanned linear space is a dense subset of $C_0^2(\mathbb{R}^2)$. Define $\forall \mu_1, \mu_2 \in \mathcal{M}(a)$,

$$\rho(\mu_1, \mu_2) = \sum_{j=1}^{\infty} 2^{-j} \left| \langle \mu_1 - \mu_2, f_j \rangle \wedge 1 \right|.$$

Then $(\mathcal{M}(a), \rho)$ is a metric space whose topology coincides with the weak topology.

Now we state a corollary of Theorem 3 (the theorem itself is given and proved in the next section).

Corollary 1. *The following (deterministic) equation*

$$\begin{aligned} \langle x_t, f \rangle &= \langle x_0, f \rangle + \int_0^t \langle x_s, (U_\delta x_s) \cdot \nabla f \rangle \, ds \\ &\quad + \sum_{j=1}^{2} \int_{\mathbb{R}^2} \int_0^t \langle x_s, \Gamma(\cdot, v) \partial_j f \rangle \, h^j(v, s) dv ds \\ &\quad + \int_0^t \langle x_s, H(s, \cdot) \cdot \nabla f \rangle \, h(s) ds \end{aligned} \tag{7}$$

has a unique solution in $\mathcal{X}_0 \equiv \{x \in C([0,T], \mathcal{M}(a)) : \ x_t \in L^2(\mathbb{R}^2), \ \forall t \in [0,T]\}$.

Proof. The conclusion follows by taking $\varepsilon = 0$ in Theorem 3. □

Denote the solution to (7) in the above corollary by $x = \gamma(h^1, h^2, h)$. Finally, we state our main result.

Theorem 1. $\{X^\varepsilon\}$, *given by* (5), *satisfies the LDP on* $C([0,T], \mathcal{M}(a))$ *with the rate function*

$$I(x) = \inf \left\{ \frac{1}{2} \sum_{j=1}^{2} \int_{\mathbb{R}^2} \int_0^T \left(h^j(v,t) \right)^2 dv dt + \frac{1}{2} \int_0^T (h(t))^2 dt : \ x = \gamma(h^1, h^2, h) \right\}.$$

3. **Existence and uniqueness of the solution to SNSE.** In this section it would be convenient to establish existence and uniqueness of the solution to an equation which is slightly more general than (5). Namely, we consider

$$\begin{cases} d \langle Y_t^\varepsilon, f \rangle &= \left\langle Y_t^\varepsilon, \frac{\varepsilon}{2} \Delta f + (U_\delta Y_t^\varepsilon) \cdot \nabla f \right\rangle dt \\ & \quad + \left\langle Y_t^\varepsilon, \frac{\varepsilon}{2} \sum_{k,\ell=1}^{2} (H(t,\cdot) H^T(t,\cdot))_{k\ell} \partial_{k\ell}^2 f \right\rangle dt \\ & \quad + \sqrt{\varepsilon} \sum_{j=1}^{2} \int_{\mathbb{R}^2} \langle Y_t^\varepsilon, \Gamma(\cdot, v) \partial_j f \rangle \, W^j(dv dt) \\ & \quad + \sqrt{\varepsilon} \langle Y_t^\varepsilon, H(t, \cdot) \cdot \nabla f \rangle \, dW^0(t) \\ & \quad + \sum_{j=1}^{2} \int_{\mathbb{R}^2} \langle Y_t^\varepsilon, \Gamma(\cdot, v) \partial_j f \rangle \, h^j(v,t) dv dt \\ & \quad + \langle Y_t^\varepsilon, H(t, \cdot) \cdot \nabla f \rangle \, h(t) dt, \ \ \forall f \in C_b^2(\mathbb{R}^2), \\ Y_0^\varepsilon &= x_0, \end{cases} \tag{8}$$

where $h^j \in L^2(\mathbb{R}^2 \times [0,T])$ and $h \in L^2([0,T])$. Assume that $x_0 \in L^2(\mathbb{R}^2) \cap \mathcal{M}(a)$.

In what follows in this section we adopt the approach of Kurtz and Xiong [16]. We will give only an outline for the arguments similar to those found in [16], but will provide details for those not in [16]. Let $\Theta = \{(v,j) : \ v \in \mathbb{R}^2, \ j \in \{1,2\}\} \cup \{\partial\}$. Define a white noise random measure on $\Theta \times \mathbb{R}_+$ by

$$W(A \times \{j\}, dt) = W^j(A, dt), \qquad W(\{\partial\}, dt) = dW^0(t), \qquad \forall A \in \mathcal{B}(\mathbb{R}^2).$$

Then the intensity measure of W is $\tilde{\mu}(d\theta)dt$, where $\tilde{\mu}$ is the Lebesgue measure on $\{(v,j):\ v\in\mathbb{R}^2,\ j\in\{1,2\}\}$ and unit mass at $\{\partial\}$. It is convenient to rewrite (8) as

$$\left\{\begin{array}{rcl} d\left\langle Y_t^\varepsilon, f\right\rangle & = & \left\langle Y_t^\varepsilon, \frac{1}{2}\sum_{k,\ell=1}^2 a_{k\ell}(t,\cdot)\partial^2_{k\ell}f + \sum_{j=1}^2 b_j(t,Y_t^\varepsilon,\cdot)\partial_j f\right\rangle dt \\ & & + \int_\Theta \left\langle Y_t^\varepsilon, \alpha(t,\cdot,\theta)\cdot\nabla f\right\rangle W(d\theta dt),\ \ \forall f\in C_b^2(\mathbb{R}^2), \\ Y_0^\varepsilon & = & x_0, \end{array}\right. \tag{9}$$

where the 2×2 matrix $a(t,q)$ is given by

$$\frac{1}{\varepsilon}a(t,q) = H(t,q)H^T(t,q) + I.$$

Also,

$$b_j(t,\eta,q) = (U_\delta\eta)_j(q) + \int_{\mathbb{R}^2}\Gamma(q,v)h^j(v,t)dv + (H(t,q))_j h(t),$$

$$\frac{1}{\sqrt{\varepsilon}}\alpha(t,q,(v,j))_k = \Gamma(q,v)\delta_{jk}$$

and

$$\frac{1}{\sqrt{\varepsilon}}\alpha(t,q,\partial) = H(t,q).$$

(Note that (9) is similar to (1.6) in [16] but, unlike in the latter case, the coefficients a_{kl}, b_j and α depend on t explicitly.) For simplicity of notation, let us put $\varepsilon = 1$ and drop the superscript ε in the rest of this section.

Consider (e_i,ξ_i), $i=1,2,\cdots$, an i.i.d. sequence of $\mathbb{R}\times\mathbb{R}^2$-valued random variables independent of $W^j, j=0,1,2$, such that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N e_i\delta_{\xi_i} = x_0,$$

where $\mathbb{E}(e_1^2)<\infty$, $\mathbb{E}(\xi_1^2)<\infty$, and $ae_1>0$ w.p.1. (a is the same constant here as in $\mathcal{M}(a)$). Note that, by the law of large numbers, $\mathbb{E}(e_i) = x_0(\mathbb{R}^2) = a$. Let us study the following system:

$$Y_t = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N e_i\delta_{x_t^i}, \tag{10}$$

where

$$x_t^i = \xi_i + \int_0^t b(s,Y_s,x_s^i)ds + \int_0^t\int_\Theta \alpha(s,x_s^i,\theta)W(d\theta ds), \qquad i=1,2,\cdots. \tag{11}$$

Lemma 1. *The system (10)–(11) has a unique solution that belongs to the space* $C\left([0,T],\mathcal{M}(a)\times(\mathbb{R}^2)^{\mathbb{N}}\right)$.

Proof. (Existence) Let

$$Y_t^N = \frac{1}{N}\sum_{i=1}^N e_i\delta_{x_t^{i,N}}, \tag{12}$$

where

$$x_t^{i,N} = \xi_i + \int_0^t b(s,Y_s^N,x_s^{i,N})ds + \int_0^t\int_\Theta \alpha(s,x_s^{i,N},\theta)W(d\theta ds), \qquad i=1,2,\cdots. \tag{13}$$

By the boundedness and Lipschitz continuity of the coefficients, (12)–(13) has a unique solution. By Itô's formula, we see that

$$\begin{aligned} d\left\langle Y_t^N, f\right\rangle &= \left\langle Y_t^N, \frac{1}{2}\sum_{k,\ell=1}^{2} a_{k\ell}(t,\cdot)\partial^2_{k\ell} f + \sum_{j=1}^{2} b_j(t, Y_t^N, \cdot)\partial_j f \right\rangle dt \\ &\quad + \int_\Theta \left\langle Y_t^N, \alpha(t,\cdot,\theta)\cdot\nabla f\right\rangle W(d\theta dt). \end{aligned}$$

By arguments similar to those found in section 5 (we prove exponential tightness there), one proves tightness of $\{(x^{i,N}, Y^N)\}$ in $C([0,T],(\mathbb{R}^2)^{\mathbb{N}}\times\mathcal{M})$, where $\mathcal{M}$ denotes the space of finite positive measures on $\mathbb{R}^2$ if $a>0$, and the space of negative signed measures if $a<0$. Any limit point will be a solution to (10)–(11). This proves the existence.

(Uniqueness) Let (x^i, Y) and $(\tilde{x}^i, \tilde{Y})$ be two solutions. Note that

$$\begin{aligned} f(t) &\equiv \mathbb{E}|x_t^i - \tilde{x}_t^i|^2 \\ &\le c_1\,\mathbb{E}\left[\int_0^t \{|a||x_s^i - \tilde{x}_s^i| + \rho_W(Y_s, \tilde{Y}_s)\}ds\right]^2 + c_2\int_0^t \mathbb{E}|x_s^i - \tilde{x}_s^i|^2 ds \end{aligned}$$

for suitable constants $c_1 = c_1(T), c_2 = c_2(T)$ and, by Schwartz inequality,

$$\begin{aligned} \rho_W(Y_s, \tilde{Y}_s)^2 &= \sup_{f\in\mathbb{B}_1}\left|\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N (e_i f(x_s^i) - e_i f(\tilde{x}_s^i))\right|^2 \\ &\le \mathbb{E}(e_1^2)\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N |x_s^i - \tilde{x}_s^i|^2. \end{aligned}$$

Then $f(t) \le K_1\int_0^t f(s)ds$ and the uniqueness follows from Gronwall's inequality. □

Applying Itô's formula to (10)–(11), it is clear that Y_t solves the SNSE (8). To obtain uniqueness, we fix an $\mathcal{M}(a)$-valued process V and consider the linear equation

$$\begin{aligned} \langle Y_t, f\rangle &= \langle x_0, f\rangle + \int_0^t \left\langle Y_s, \frac{1}{2}\sum_{k,\ell=1}^{2} a_{k\ell}(s,\cdot)\partial^2_{k\ell} f + \sum_{j=1}^{2}\tilde{b}_j(s,\cdot)\partial_j f\right\rangle ds \\ &\quad + \int_0^t\int_\Theta \langle Y_s, \alpha(s,\cdot,\theta)\cdot\nabla f\rangle W(d\theta ds), \end{aligned} \tag{14}$$

where $\tilde{b}_j(s,q) = b_j(s, V_s, q)$. Let $G_\lambda(r) = (2\pi\lambda)^{-1}\exp\left(-|r|^2/2\lambda\right)$ and consider the following process $Y_s^\lambda(r) \equiv (T_\lambda[Y_s])(r) \equiv \int_{\mathbb{R}^2} G_\lambda(r-q)Y_s(dq)$. Then $Y_s^\lambda \in L^2(\mathbb{R}^2)$ and

$$\begin{aligned} &\left\langle Y_t^\lambda, f\right\rangle \\ &= \langle T_\lambda x_0, f\rangle + \int_0^t \left\langle \frac{1}{2}\sum_{k,\ell=1}^{2}\partial^2_{k\ell}(T_\lambda[a_{k\ell}(s,\cdot)Y_s]) - \sum_{j=1}^{2}\partial_j(T_\lambda[\tilde{b}_j(s,\cdot)Y_s]), f\right\rangle ds \\ &\quad - \int_0^t\int_\Theta \langle\nabla\cdot(T_\lambda[\alpha(s,\cdot,\theta)Y_s]), f\rangle W(d\theta ds). \end{aligned}$$

By the same argument as in the proof of Theorem 3.2 in [16], we have

$$\mathbb{E}\|Y_t^\lambda\|^2_{L^2(\mathbb{R}^2)} \le \|T_\lambda x_0\|^2_{L^2(\mathbb{R}^2)} + \int_0^t K_1(s)\mathbb{E}(\|T_\lambda[Y_s]\|^2_{L^2(\mathbb{R}^2)})ds \tag{15}$$

where $K_1 \in L^2([0,T])$.

Theorem 2. *Equation (8) has an $L^2(\mathbb{R}^2)$-valued solution.*

Proof. Let V_t be the solution of (10)–(11) (as we mentioned earlier, it also solves (8)). Define

$$V_t = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N e_i \delta_{x_t^i}$$

where

$$x_t^i = \xi_i + \int_0^t b(s, V_s, x_s^i)ds + \int_0^t \int_\Theta \alpha(s, x_s^i, \theta)W(d\theta ds), \qquad i = 1, 2, \cdots.$$

Applying Itô's formula, we see that V solves (14). By (15), it follows then that

$$\mathbb{E}\|V_t^\lambda\|^2_{L^2(\mathbb{R}^2)} \le \|T_\lambda x_0\|^2_{L^2(\mathbb{R}^2)} + \int_0^t K_1(s)\mathbb{E}(\|V_s^\lambda\|^2_{L^2(\mathbb{R}^2)})ds.$$

By Gronwall's inequality, we have

$$\mathbb{E}\|V_t^\lambda\|^2_{L^2(\mathbb{R}^2)} \le K_2\|T_\lambda x_0\|^2_{L^2(\mathbb{R}^2)}.$$

Let $\{\phi_j\}$ be a complete orthonormal system in $L^2(\mathbb{R}^2)$ such that $\phi_j \in C_b(\mathbb{R}^2)$. Then, upon taking $\lambda \to 0$ and applying Fatou's lemma, we see that

$$\begin{aligned} \mathbb{E}\left[\sum_j \langle V_t, \phi_j\rangle^2\right] &= \mathbb{E}\left[\sum_j \lim_{\lambda\to 0}\langle V_t^\lambda, \phi_j\rangle^2\right] \\ \le \quad \liminf_{\lambda\to 0} \mathbb{E}\|V_t^\lambda\|^2_{L^2(\mathbb{R}^2)} &\le K_2\|x_0\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Therefore, $V_t \in L^2(\mathbb{R}^2)$ a.s. and $\mathbb{E}\|V_t\|^2_{L^2(\mathbb{R}^2)} < \infty$. □

Theorem 3. *Equation (8) has at most one $L^2(\mathbb{R}^2)$-valued solution.*

Proof. Suppose $\hat{V}_t$ is another $L^2(\mathbb{R}^2)$-valued solution of (8). Define

$$x_t^i = \xi_i + \int_0^t b(s, \hat{V}_s, x_s^i)ds + \int_0^t \int_\Theta \alpha(s, x_s^i, \theta)W(d\theta ds), \qquad i = 1, 2, \cdots$$

and

$$\hat{Y}_t := \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N e_i \delta_{x_t^i}.$$

Let $\widehat{(14)}$ denote the linear equation (14) with V replaced by $\hat{V}$. Then $\hat{Y}_t$ solves $\widehat{(14)}$. Since $\hat{V}_t$ also solves $\widehat{(14)}$, then $\hat{Y}_t - \hat{V}_t$ solves equation $\widehat{(14)}$ with x_0 replaced by 0. Hence, as in (15), it follows that

$$\mathbb{E}\|\hat{Y}_t^\lambda - \hat{V}_t^\lambda\|^2_{L^2(\mathbb{R}^2)} \le \int_0^t K_1(s)\mathbb{E}\left(\|T_\lambda(|\hat{Y}_s - \hat{V}_s|)\|^2_{L^2(\mathbb{R}^2)}\right)ds.$$

Taking $\lambda \to 0$, we deduce that

$$\mathbb{E}\|\hat{Y}_t - \hat{V}_t\|^2_{L^2(\mathbb{R}^2)} \le \int_0^t K_1(s)\mathbb{E}\|\hat{Y}_s - \hat{V}_s\|^2_{L^2(\mathbb{R}^2)}ds.$$

Therefore, Gronwall's inequality and the sample path continuity imply that $P(\hat{Y}_t = \hat{V}_t;\ 0 \le t \le T) = 1$, which (upon taking into account Lemma 1) implies the required uniqueness of the solution to (8). □

Let $\mathcal{M}_0$ be the collection of all finite signed measures in $\mathcal{M}(a)$ which have a square-integrable density with respect to the Lebesgue measure. The topology of $\mathcal{M}_0$ is the subspace topology of $\mathcal{M}(a)$.

Corollary 2. *The SNSE (5) has a unique solution in $\mathcal{M}_0$.*

Proof. The conclusion of this corollary follows from Theorems 2 and 3 by taking h^j and h to be zero in (8). □

4. **Exponential continuity.** Let $\mathcal{X} = C([0,T], \mathcal{M}(a) \cap L^2(\mathbb{R}^2))$. Let $\mathcal{D}$ be the Banach space $C_b^2(\mathbb{R}^2)$ with norm

$$\|f\| = \sup_{r \in \mathbb{R}^2; i,j \in \{1,2\}} (|f(r)| + |\partial_i f(r)| + |\partial_{ij} f(r)|).$$

Let $\mathcal{H}_0 \subset \mathcal{D}$ be such that $(\iota, \mathcal{H}_0, \mathcal{D})$ is an abstract Wiener space. Then $\mathcal{M}(a) \subset C_b(\mathbb{R}^2)' \subset \mathcal{H}_0'$. Let $\mathcal{Y} = C([0,T], \mathcal{H}_0')$ and $\mathcal{Z} = C([0,T], \mathbb{B})$ where $\mathbb{B}$ is a Banach space such that $(\iota, L^2(\Theta, \tilde{\mu}), \mathbb{B})$ is an abstract Wiener space. Identify the white noise random measure W with an $L^2(\Theta, \tilde{\mu})$-cylindrical Brownian motion (cf. [13]) and, still denote it by W. Let $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ be the collection of all continuous processes on $[0,T]$ taking values in $\mathcal{M}(a)$ and $\mathcal{H}_0'$ respectively. Let $\mathcal{A}_{\mathcal{Z}}$ be the collection of all continuous $\mathbb{B}$-valued semimartingales. Also, given a continuous martingale M, let $\ll M, M \gg$ denote its quadratic variation process. Consider

$$\left\langle F(X^\varepsilon, \sqrt{\varepsilon} W)_t, f \right\rangle := \left\langle F_1(X^\varepsilon, \sqrt{\varepsilon} W)_t, f \right\rangle + \left\langle F_2(X^\varepsilon, \sqrt{\varepsilon} W)_t, f \right\rangle, \tag{16}$$

where we put

$$\begin{aligned}
& \left\langle F_1(X, \sqrt{\varepsilon} W)_t, f \right\rangle \\
:= \ & \langle x_0, f \rangle + \int_0^t \langle X_s, (U_\delta X_s) \cdot \nabla f \rangle \, ds \\
& + \frac{1}{2} \sum_{j=1}^{2} \int_0^t \left\langle X_s, \partial_{jj}^2 f \right\rangle d \ll \int_0^{\cdot} \int_{\mathbb{R}^2} \Gamma(\overrightarrow{0}, v) \sqrt{\varepsilon} W^j (d\tau dv), \\
& \int_0^{\cdot} \int_{\mathbb{R}^2} \Gamma(\overrightarrow{0}, v) \sqrt{\varepsilon} W^j (d\tau dv) \gg_s \\
& + \int_0^t \left\langle X_s, \frac{1}{2} \sum_{k,\ell=1}^{2} (H(s,\cdot) H^T(s,\cdot))_{k\ell} \partial_{k\ell}^2 f \right\rangle d \ll \sqrt{\varepsilon} W^0, \sqrt{\varepsilon} W^0 \gg_s \qquad (17) \\
= \ & \langle x_0, f \rangle + \int_0^t \left\langle X_s, \frac{\varepsilon}{2} \Delta f + (U_\delta X_s) \cdot \nabla f + \frac{\varepsilon}{2} \sum_{k,\ell=1}^{2} (H(s,\cdot) H^T(s,\cdot))_{k\ell} \partial_{k\ell}^2 f \right\rangle ds, \qquad (18)
\end{aligned}$$

(where (18) follows from (17) since, by (A2), $\int_{\mathbb{R}^2}\left(\Gamma(\overrightarrow{0},v)\right)^2 dv = 1$, where $\overrightarrow{0} = (0,0)'$) and let

$$\begin{aligned}\left\langle F_2(X,\sqrt{\varepsilon}W)_t, f\right\rangle \quad &:= \quad \sum_{j=1}^{2}\int_0^t\int_{\mathbb{R}^2}\langle X_s,\Gamma(\cdot,v)\partial_j f\rangle\sqrt{\varepsilon}W^j(dvds)\\ &\quad + \int_0^t \langle X_s, H(s,\cdot)\cdot\nabla f\rangle\, d(\sqrt{\varepsilon}W^0(s)). \qquad (19)\end{aligned}$$

The following proposition is useful for establishing exponential continuity of F in this section and for verification of exponential tightness of $\{X^\varepsilon\}$ in the next section. The proof of the proposition follows from the same argument as in Lemma 10.2.2 of [13].

Proposition 2. *Let W be a Brownian sheet on $\Theta\times\mathbb{R}_+$ with intensity $\tilde{\mu}(d\theta)dt$. Suppose that $f:\ [0,T]\times\Theta\times\Omega\to\mathcal{H}$ is a predictable map such that*

$$K_2 \equiv \sup_{0\le t\le T,\ \omega\in\Omega}\int_\Theta |f(t,\theta,\omega)|^2_{\mathcal{H}}\,\tilde{\mu}(d\theta) < \infty$$

where $\mathcal{H}$ is a Hilbert space. Let $\eta_t = \int_0^t\int_\Theta f(s,\theta)W(d\theta ds)$ and $0<\alpha<1/2$. Fix arbitrary $M>0$. Then there exists $\varepsilon_0=\varepsilon_0(M,K_2)$ such that

$$\varepsilon\log\mathbb{P}\left([\sqrt{\varepsilon}\eta]_\alpha > M\right) \le -\left(\frac{M}{K_3}-\sqrt{\varepsilon}\right)^2 + \varepsilon K_4$$

for all $\varepsilon\in(0,\varepsilon_0)$, where

$$[\eta]_\alpha = \sup_{0\le s<t\le T}\frac{|\eta_t-\eta_s|_{\mathcal{H}}}{|t-s|^\alpha}$$

is the Hölder norm and K_3 and K_4 are two constants depending on K_2 and T.

Proposition 3. *F is exponentially continuous.*

Proof. For F_1 and F_2 given by (17) – (19), note first that

$$\sup_{0\le t\le T}\sup_{\|f\|\le 1}\left|\langle F_1(X^1,\sqrt{\varepsilon}W)_t - F_1(X^2,\sqrt{\varepsilon}W)_t, f\rangle\right| \le c\sup_{0\le t\le T}\rho_W(X^1_t,X^2_t)$$

for some constant $c=c(T,|a|)$, for all $0\le\varepsilon\le 1$. Similarly, if we let $\tilde{F}(s,(v,j))f := \left\langle X^1_s - X^2_s, \Gamma(\cdot,v)\partial_j f\right\rangle$ and $\tilde{F}(s,\{\partial\})f := \left\langle X^1_s - X^2_s, H(s,\cdot)\cdot\nabla f\right\rangle$, one obtains that

$$\sup_{0\le s\le T}\int_\Theta\left|\tilde{F}(s,\theta)\right|^2_{\mathcal{H}'_0}\tilde{\mu}(d\theta) \le c'\sup_{0\le s\le T}\rho^2_W(X^1_s,X^2_s)$$

for some constant c'. Therefore, for any δ_1, there exists δ' such that $d_{\mathcal{X}}(X^1,X^2)<\delta'$ implies that $d_{\mathcal{Y}}(F_1(X^1,\sqrt{\varepsilon}W),F_1(X^2,\sqrt{\varepsilon}W))<\delta_1$ for all $0\le\varepsilon\le 1$ and

$$\sup_{0\le s\le T}\int_\Theta\left|\tilde{F}(s,\theta)\right|^2_{\mathcal{H}'_0}\tilde{\mu}(d\theta)\le\delta_1^2.$$

Note that

$$\|F_2(X^1,\sqrt{\varepsilon}W)-F_2(X^2,\sqrt{\varepsilon}W)\|_{\mathcal{Y}} \le T^\alpha\left[\sqrt{\varepsilon}\int_0^\cdot\int_\Theta\tilde{F}(s,\theta)W(d\theta ds)\right]_\alpha.$$

Therefore, for all $\delta'' > 0$,

$$\begin{aligned}
& \varepsilon \log \mathbb{P}\left(d_{\mathcal{X}}(X^1, X^2) < \delta', d_{\mathcal{Y}}(F(X^1, \sqrt{\varepsilon}W), F(X^2, \sqrt{\varepsilon}W)) > \delta''\right) \\
\leq\ & \varepsilon \log \mathbb{P}\left(\left[\sqrt{\varepsilon}\int_0^{\cdot}\int_\Theta \delta_1^{-1}\tilde{F}(s,\theta)W(d\theta ds)\right]_\alpha \geq \frac{\delta''-\delta_1}{\delta_1 T^\alpha}\right) \\
\leq\ & -\left(\frac{\delta''-\delta_1}{\delta_1 K_3 T^\alpha} - \sqrt{\varepsilon}\right)^2 + \varepsilon K_4
\end{aligned}$$

for all sufficiently small $\varepsilon > 0$. The conclusion then follows by taking $\varepsilon \downarrow 0$ and suitably choosing δ_1. □

5. **Exponential tightness.** In this section, we verify the exponential tightness of $\{X^\varepsilon\}$ on $C([0,T], \mathcal{M}(a))$.

Lemma 2. *$\forall\ L > 0$, $\forall\ n \in \mathbb{N}$, $\exists\ R_n > 0$ such that*

$$\varepsilon \log \mathbb{P}\left(\sup_{0\leq t\leq T} |Y_t^\varepsilon|((\mathbb{U}_{R_n})^c) > \frac{1}{n}\right) \leq -nL$$

where $\mathbb{U}_R = \{r \in \mathbb{R}^2 :\ |r| \leq R\}$.

Proof. Application of Itô's formula to (10)–(11) (restore ε there) yields that

$$\begin{aligned}
1 + |x_t^i|^2 &= 1 + |\xi_i|^2 + 2\int_0^t x_s^i \cdot b(s, Y_s, x_s^i)ds \\
&\quad + \int_0^t \int_\Theta |\alpha(s, x_s^i, \theta)|^2 \tilde{\mu}(d\theta)ds + 2\int_0^t \int_\Theta x_s^i \cdot \alpha(s, x_s^i, \theta)W(d\theta ds),
\end{aligned}$$

and, by Itô's formula again,

$$\begin{aligned}
& \sqrt{1 + |x_t^i|^2} \\
=\ & \sqrt{1 + |\xi_i|^2} + \int_0^t \frac{x_s^i \cdot b(s, Y_s, x_s^i)}{\sqrt{1 + |x_s^i|^2}}ds + \int_0^t \int_\Theta \frac{x_s^i \cdot \alpha(s, x_s^i, \theta)}{\sqrt{1 + |x_s^i|^2}}W(d\theta ds) \\
& + \frac{1}{2}\int_0^t \int_\Theta \frac{|\alpha(s, x_s^i, \theta)|^2}{\sqrt{1 + |x_s^i|^2}}\tilde{\mu}(d\theta)ds - \frac{1}{2}\int_0^t \int_\Theta \frac{(x_s^i \cdot \alpha(s, x_s^i, \theta))^2}{\sqrt{(1 + |x_s^i|^2)^3}}\tilde{\mu}(d\theta)ds.
\end{aligned}$$

Then

$$\begin{aligned}
\sqrt{1 + |x_t^i|^2} &= \sqrt{1 + |\xi_i|^2} + \int_0^t \frac{x_s^i \cdot b(s, Y_s, x_s^i)}{\sqrt{1 + |x_s^i|^2}}ds \\
&\quad + \sqrt{\varepsilon}\int_0^t \int_{\mathbb{R}^2} \Gamma(x_s^i, v)\frac{x_s^i}{\sqrt{1 + |x_s^i|^2}} \cdot \begin{pmatrix} W^1 \\ W^2 \end{pmatrix}(dvds) \\
&\quad + \sqrt{\varepsilon}\int_0^t \frac{x_s^i \cdot H(s, x_s^i)}{\sqrt{1 + |x_s^i|^2}}dW^0(s) \\
&\quad + \varepsilon\int_0^t \frac{2 + |H(s, x_s^i)|^2}{2\sqrt{1 + |x_s^i|^2}}ds - \varepsilon\int_0^t \frac{|x_s^i|^2 + (x_s^i \cdot H(s, x_s^i))^2}{2\sqrt{(1 + |x_s^i|^2)^3}}ds.
\end{aligned}$$

Let $f(r) = \sqrt{1+|r|^2}$. Then the following representation holds:

$$\begin{aligned}\langle |Y_t^\varepsilon|, f\rangle &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N |e_i|\sqrt{1+|x_t^i|^2} \\ &= \langle |x_0|, f\rangle + \int_0^t \beta_s ds + \sqrt{\varepsilon}\int_0^t \sigma_s d\tilde{B}_s + \varepsilon\int_0^t \rho_s ds\end{aligned}$$

where $\tilde{B}$ is a Brownian motion, σ_s and ρ_s are bounded by some constant $\tilde{C}$ and $\int_0^T |\beta_s| ds \le \tilde{C}$. By an argument similar to the one in [13] (p. 313), based on viewing $\int_0^t \sigma_s d\tilde{B}_s$ as a time-changed Wiener process and subsequent application of Fernique's theorem (cf. Kuo [15]), there exists a constant $c_1 > 0$ such that

$$\tilde{K} \equiv \mathbb{E}\exp\left(\sup_{0\le t\le T} c_1 \left|\int_0^t \sigma_s d\tilde{B}_s\right|^2\right) < \infty.$$

So, for all $M > \langle |x_0|, f\rangle + \tilde{C}(1+\varepsilon T)$,

$$\begin{aligned}&\varepsilon \log \mathbb{P}\left(\sup_{0\le t\le T} \langle |Y_t^\varepsilon|, f\rangle > M\right) \\ \le\ & \varepsilon \log \mathbb{P}\left(\sup_{0\le t\le T} \left|\int_0^t \sigma_s d\tilde{B}_s\right|^2 > \frac{\tilde{M}}{\varepsilon}\right) \\ \le\ & \varepsilon \log\left(\tilde{K} e^{-\frac{c_1\tilde{M}}{\varepsilon}}\right) = \varepsilon\log\tilde{K} - c_1\tilde{M},\end{aligned}$$

where $\tilde{M} = \left(M - \langle |x_0|, f\rangle - \tilde{C} - \varepsilon\tilde{C}T\right)^2$. Hence, given an arbitrary $L > 0$, the following inequality holds:

$$\begin{aligned}&\varepsilon \log \mathbb{P}\left(\sup_{0\le t\le T} |Y_t^\varepsilon|\left((\mathbb{U}_R)^c\right) > \frac{1}{n}\right) \\ \le\ & \varepsilon \log \mathbb{P}\left(\sup_{0\le t\le T} \langle |Y_t^\varepsilon|, f\rangle > \frac{1}{n}\sqrt{1+R^2}\right) \\ \le\ & \varepsilon\log\tilde{K} - c_1\left(\frac{1}{n}\sqrt{1+R^2} - \langle |x_0|, f\rangle - \tilde{C}(1+\varepsilon T)\right)^2 \le -nL\end{aligned}$$

for some R depending on n. □

Lemma 3. *Let $0 < \alpha < 1/2$. Then $\forall L > 0$, $\forall j \in \mathbb{N}$, $\exists M_j$ such that*

$$\varepsilon \log \mathbb{P}\left([\langle Y^\varepsilon, f_j\rangle]_\alpha > M_j\right) \le -jL.$$

Proof. Note that in equation (9), we have

$$\begin{aligned}&\left[\int_0^\cdot \left\langle Y_s^\varepsilon, \sum_{k,\ell=1}^2 a_{k\ell}(s,\cdot)\partial^2_{k\ell} f_j + \sum_{i=1}^2 b_i(s, Y_s^\varepsilon, \cdot)\partial_i f_j\right\rangle ds\right]_\alpha \\ \le\ & K_5 \equiv K_5(j, |a|, T, \alpha, C)\end{aligned}$$

for all $0 < \varepsilon < 1$; and

$$\int_\Theta \left\langle Y_t^\varepsilon, \frac{1}{\sqrt{\varepsilon}}\alpha(t,\cdot,\theta)\cdot\nabla f_j\right\rangle^2 \tilde{\mu}(d\theta) \le K_6 \equiv K_6(j, |a|, C) < \infty.$$

By Proposition 2, for all $M_j > K_5$ we have that

$$\begin{aligned} & \varepsilon \log \mathbb{P}\left([\langle Y^\varepsilon, f_j\rangle]_\alpha > M_j\right) \\ \leq\ & \varepsilon \log \mathbb{P}\left(\left[\sqrt{\varepsilon}\int_0^\cdot \int_\Theta \left\langle Y_s^\varepsilon, \frac{1}{\sqrt{\varepsilon}}\alpha(s,\cdot,\theta)\cdot\nabla f_j\right\rangle W(d\theta ds)\right]_\alpha > M_j - K_5\right) \\ \leq\ & -\left(\frac{M_j - K_5}{K_3} - \sqrt{\varepsilon}\right)^2 + \varepsilon K_4 \end{aligned}$$

for all sufficiently small $\varepsilon > 0$. The conclusion of the lemma then follows by suitably choosing M_j. □

Proposition 4. *$\{Y^\varepsilon\}$ is exponentially tight. As a consequence, $\{X^\varepsilon\}$ is exponentially tight.*

Proof. Fix arbitrary $L > 0$ and choose $\{R_n\}$ and $\{M_j\}$ as in Lemma 2 and Lemma 3. Let

$$\mathbb{K} = \left\{ \mu \in C([0,T], \mathcal{M}(a)) : \begin{array}{l} \sup_{0\leq t\leq T} |\mu_t|\left((\mathbb{U}_{R_n})^c\right) \leq \frac{1}{n},\ \forall n \geq 1; \\ [\langle \mu, f_j\rangle]_\alpha \leq M_j,\ \forall j \geq 1 \end{array} \right\}.$$

Then $\mathbb{K}$ is a compact set and

$$\begin{aligned} \mathbb{P}(Y^\varepsilon \notin \mathbb{K}) &\leq \sum_{n=1}^\infty \mathbb{P}\left(\sup_{0\leq t\leq T} |Y_t^\varepsilon|\left((\mathbb{U}_{R_n})^c\right) > \frac{1}{n}\right) + \sum_{j=1}^\infty \mathbb{P}\left([\langle Y^\varepsilon, f_j\rangle]_\alpha > M_j\right) \\ &\leq \sum_{n=1}^\infty e^{-nL/\varepsilon} + \sum_{j=1}^\infty e^{-jL/\varepsilon} = \frac{2}{e^{L/\varepsilon} - 1}. \end{aligned}$$

Thus $\forall \varepsilon \leq L$, $\mathbb{P}(Y^\varepsilon \notin \mathbb{K}) \leq 4e^{-L/\varepsilon}$, which implies that

$$\varepsilon \log \mathbb{P}(Y^\varepsilon \notin \mathbb{K}) \leq \varepsilon \log 4 - L$$

for all sufficiently small $\varepsilon > 0$. Exponential tightness of $\{Y^\varepsilon\}$ then follows at once. The conclusion for X^ε is obtained by taking h^j and h as 0. □

6. **Proof of Theorem 1.**

In this section, we finish the proof of Theorem 1 by verifying conditions (L3-L6).

Proof. Define $\mathcal{H}$ to be the collection of all $L^2(\Theta, \tilde{\mu})$-valued absolutely continuous maps ℓ on $[0,T]$ such that $\dot{\ell} \in L^2([0,T]; L^2(\Theta, \tilde{\mu}))$. Then, for $\ell \in \mathcal{H}$, there exist functions $h^j \in L^2([0,T]\times\mathbb{R}^2)$, $j = 1,\ 2$, and $h \in L^2(\mathbb{R}^2)$ such that $\dot{\ell}_t(v,j) = h^j(t,v)$ and $\dot{\ell}_t(\{\partial\}) = h(t)$. Then (L3) follows from Girsanov's formula with $\xi_\ell^\varepsilon = \xi_\ell/\sqrt{\varepsilon}$, where

$$\xi_\ell = \sum_{j=1}^2 \int_0^t \int_{\mathbb{R}^2} h^j(s,v) W^j(dvds) + \int_0^t h(s) dW^0(s).$$

(L4) follows from Corollary 1. Moreover, by Proposition 4, we see that $\{Y^\varepsilon\}$ is tight. Taking $\varepsilon \to 0$, it is easy to see from (8) that the limit point solves (7) which has a unique solution $x = \gamma(h^1, h^2, h)$. This proves (L5). Finally, $\forall x \in \mathcal{X}$, let $X^{x,\varepsilon}$

be given by

$$\begin{cases} d\left\langle X_t^{x,\varepsilon}, f\right\rangle &= \left\langle x_t, \frac{\varepsilon}{2}\Delta f + (U_\delta x_t)\cdot\nabla f\right\rangle dt \\ & \quad + \left\langle x_t, \frac{\varepsilon}{2}\sum_{k,\ell=1}^2 (H(t,\cdot)H^T(t,\cdot))_{k\ell}\partial^2_{k\ell} f\right\rangle dt \\ & \quad + \sqrt{\varepsilon}\sum_{j=1}^2 \int_{\mathbb{R}^2} \left\langle x_t, \Gamma(\cdot,v)\partial_j f\right\rangle W^j(dvdt) \\ & \quad + \sqrt{\varepsilon}\left\langle x_t, H(t,\cdot)\cdot\nabla f\right\rangle dW^0(t), \\ X_0^{x,\varepsilon} &= x_0. \end{cases}$$

Then $X^{x,\varepsilon}$ is a Gaussian process. Its LDP follows from a general result of Kallianpur and Oodaira [12]. Theorem 1 then follows from Proposition 1. □

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