Stochastic transfer principle for multiple integrals with respect to Gaussian random fields with applications

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Abstract. A stochastic transfer principle, that relates multiple Itô-type integrals with respect to a general Gaussian random field X, to multiple stochastic integrals defined with respect to a Volterra-type field Y with respect to X of the form $Y_{t_1,\dots,t_d} = \int_0^{t_1} \cdots \int_0^{t_d} \prod_{i=1}^d K_i(t_i, s_i) dX_{s_1,\dots,s_d}$, is established. As an illustration, the result is applied to deduce the product formula for $I_n^Y(f) I_m^X(g)$ and the independence criterion for $I_n^Y(f)$, $I_m^X(g)$; in the special case when Y is a Volterra-type field with respect to a persistent fractional Brownian sheet, the continuity (for the topology of total variation on the space of signed measures) of the law of integrals $I_n^Y(f)$ with respect to the kernel f is also deduced.

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1 Introduction

We are interested in establishing a relationship between multiple stochastic Itô-type integrals with respect to a given d-parameter ($d \in \mathbb{N}$) Gaussian random field X and multiple stochastic Itô-type integrals with respect to Y, where Y is a Volterra-type field with respect to X, i.e. $Y_{t^{(1)},\dots,t^{(d)}} = \int_{\mathbb{T}} \prod_{j=1}^d K_j(t^{(j)}, s^{(j)}) dX_{s^{(1)},\dots,s^{(d)}}$, where precise conditions on kernels $K_j(\cdot,\cdot)$ are given in the next section. The special case of $d = 1$ and $(X, Y) = (W, B^H)$, where W is a standard Wiener process and B^H is a standard fractional Brownian motion with Hurst index $H \in (0, 1)$, was studied in [4], and a stochastic transfer principle for multiple Itô-type integrals $I_n^{B^H}(\cdot)$ and $I_n^W(\cdot)$ was established. The goal of the present paper is to obtain a general stochastic transfer principle in the case of $d > 1$ and X being a general Gaussian field (i.e. X can be nonstationary, may have dependent increments, need not be a martingale nor a Markov process).

The paper is organized as follows. The remainder of the present section is devoted to discussion of function spaces associated with multiple Itô-type integrals with respect to general Gaussian random fields. Section 2 contains the main result of the paper, namely, the stochastic transfer principle, plus a number of corollaries pertaining to the product

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formula for $I_n^Y(f)I_m^X(g)$, independence criterion for $I_n^Y(f)$ and $I_m^X(g)$, and the continuity of the law (in the topology of total variation) of the multiple Itô integral, defined for a Volterra random field with respect to a persistent fractional Brownian sheet, in terms of its integrand.

Consider an arbitrary Gaussian random field $X = (X_{\underline{t}}, \underline{t} \in \mathbb{T})$, defined on a probability space $(\Omega, \mathcal{F}, P)$, with zero mean and covariance function $\gamma(\underline{t}, \underline{s})$, and parametrized by $\mathbb{T}$, where $\mathbb{T} = [0, \infty)^d$ or $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_d]$. Assume that γ is of bounded variation on bounded subsets of $\mathbb{T}^2$, and $X_{(t^{(1)}, \ldots, 0, \ldots, t^{(d)})} = 0$ almost surely. Consider the set $\mathcal{E}^n$ of all step functions f on $\mathbb{T}^n$ of the form

$$f(\underline{t}_1, \ldots, \underline{t}_n) = \sum_{i_1, \ldots, i_n = 1}^{N} f_{i_1, \ldots, i_n} 1_{A_{i_1} \times \cdots \times A_{i_n}}(\underline{t}_1, \ldots, \underline{t}_n),$$

where each $A_{i_k} = (a_{i_k}^{(1)}, b_{i_k}^{(1)}] \times \cdots \times (a_{i_k}^{(d)}, b_{i_k}^{(d)}]$ is a bounded rectangle in $\mathbb{T}$. For all $f, g \in \mathcal{E}^n$ define

$$\langle f, g \rangle_{L^2_\gamma(\mathbb{T}^n)} = \int_{\mathbb{T}^2} \cdots \int_{\mathbb{T}^2} f(\underline{t}_1, \ldots, \underline{t}_n) g(\underline{s}_1, \ldots, \underline{s}_n) \gamma(d\underline{t}_1, d\underline{s}_1) \ldots \gamma(d\underline{t}_n, d\underline{s}_n),$$

where the integral is taken in the Riemann-Stieltjes sense. Let $\Lambda^2_\gamma(\mathbb{T}^n)$ be the completion of $(\mathcal{E}^n, \langle \cdot, \cdot \rangle_{L^2_\gamma(\mathbb{T}^n)})$. Note that, in general, the space $\Lambda^2_\gamma(\mathbb{T}^n)$ consists of generalized functions. Define $L^2_\gamma(\mathbb{T}^n)$ to be the set of all measurable functions f on $\mathbb{T}^n$ such that the following Lebesgue integrals are finite

$$\int_{\mathbb{T}^{2n}} |f(\underline{t}_1, \cdots, \underline{t}_n) f(\underline{s}_1, \cdots, \underline{s}_n)| \, |\gamma|(d\underline{t}_1, d\underline{s}_1) \ldots |\gamma|(d\underline{t}_n, d\underline{s}_n) < \infty, \text{ and}$$

$$\int_{\mathbb{T}^{2n}} |f(\underline{t}_1, \cdots, \underline{t}_n)| \, 1_{A_1 \times \cdots \times A_n}(\underline{s}_1, \cdots, \underline{s}_n) |\gamma|(d\underline{t}_1, d\underline{s}_1) \ldots |\gamma|(d\underline{t}_n, d\underline{s}_n) < \infty$$

for all $A_1 \times \cdots \times A_n \in \mathbb{T}^n$, with each $A_i \subset \mathbb{T}$ being a bounded rectangle. Then, by an argument similar to the one given in [1] for the case $d = 1$, it follows that $L^2_\gamma(\mathbb{T}^n)$ is dense in $\Lambda^2_\gamma(\mathbb{T}^n)$, and $\forall f, g \in L^2_\gamma(\mathbb{T}^n)$ such that

$$\int_{\mathbb{T}^{2n}} |f(\underline{t}_1, ..., \underline{t}_n) g(\underline{s}_1, ..., \underline{s}_n)| \, |\gamma|(d\underline{t}_1, d\underline{s}_1) ... |\gamma|(d\underline{t}_n, d\underline{s}_n) < \infty,$$

their scalar product is equal to

$$\langle f, g \rangle_{\gamma, n} = \int_{\mathbb{T}^{2n}} f(\underline{t}_1, \cdots, \underline{t}_n) g(\underline{s}_1, \cdots, \underline{s}_n) \gamma(d\underline{t}_1, d\underline{s}_1) ... \gamma(d\underline{t}_n, d\underline{s}_n).$$

Then, for all $f \in L^2_\gamma(\mathbb{T}^n)$, the Itô-type integral of order n of f with respect to Gaussian random field X, denoted by $I_n^X(f)$, is defined via a standard extension of the arguments given in [1], [2], and its properties are analogous to the properties of Itô-type integrals in the one-parameter case, which can be found in those papers.

2 Transfer principle

Let $X = (X_{\underline{t}}, \underline{t} \in \mathbb{T})$ be a Gaussian random field on a complete probability space $(\Omega, \mathcal{F}, P)$ with zero mean and covariance function $\gamma(\underline{t}, \underline{s})$, parametrized by $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_d]$. Assume that $\gamma(\cdot, \cdot)$ is a function of bounded Vitali variation on $\mathbb{T}^2$ and $X_{(t^{(1)},\ldots,0,\ldots,t^{(d)})} = 0$ almost surely. Suppose $Y = (Y_{\underline{t}})_{\underline{t}\in\mathbb{T}}$ is a continuous Gaussian random field of the form

$$Y_{\underline{t}} = \int_{\mathbb{T}} K(\underline{t}, \underline{s}) dX_{\underline{s}}, \tag{1}$$

where, for all $\underline{t} = (t^{(1)}, \ldots, t^{(d)})$, $\underline{s} = (s^{(1)}, \ldots, s^{(d)})$, the kernel K is of the form

$$K(\underline{t}, \underline{s}) = \prod_{i=1}^{d} K_i\left(t^{(i)}, s^{(i)}\right), \quad \text{with} \quad K_i\left(t^{(i)}, s^{(i)}\right) = 0 \quad \text{for all} \quad s^{(i)} \geq t^{(i)}, \tag{2}$$

and where, for all $(\underline{a}, \underline{b}] = (a^{(1)}, b^{(1)}] \times \cdots \times (a^{(d)}, b^{(d)}] \subset \mathbb{T}$,

$$\sup_{\underline{t},\underline{s}\in\mathbb{T}} \int_{\mathbb{T}^2} |K(\underline{t}, \underline{u})K(\underline{s}, \underline{v})|\, |\gamma|(d\underline{u}, d\underline{v}) < \infty \quad \text{and} \quad \sup_{\underline{t}\in\mathbb{T}} \int_{\mathbb{T}^2} |K(\underline{t}, \underline{u})| 1_{(\underline{a},\underline{b}]}(\underline{v}) |\gamma|(d\underline{u}, d\underline{v}) < \infty, \tag{3}$$

where $|\gamma|$ is the total variation measure and the two integrals are taken in the Lebesgue sense. Let us call Y a Volterra type random field with respect to X with kernel $K(\cdot, \cdot)$ if Y has a representation (1)–(3). Then Y is a zero mean Gaussian random field with the covariance function

$$R(\underline{t}, \underline{s}) = \mathbb{E}(Y_{\underline{t}}, Y_{\underline{s}}) = \int_{\mathbb{T}^2} K(\underline{t}, \underline{u})K(\underline{s}, \underline{v})\gamma(d\underline{u}, d\underline{v}),$$

where the last integral is understood in the Riemann-Stieltjes sense. Let us denote by $\mathcal{H}_Y$ the canonical space of Y, that is $\mathcal{H}_Y$ is the closure of the linear space of linear combinations of indicator functions $1_{[0,t^{(1)}]\times\cdots\times[0,t^{(d)}]}$, $(t^{(1)}, \ldots, t^{(d)}) \in \mathbb{T}$, with respect to the scalar product defined by

$$\langle 1_{[0,t^{(1)}]\times\cdots\times[0,t^{(d)}]}, 1_{[0,s^{(1)}]\times\cdots\times[0,s^{(d)}]}\rangle_{\mathcal{H}_Y} = R(\underline{t}, \underline{s}).$$

Note that $\mathcal{H}_Y$ is isometric to the closure of the linear span of functions $\{K(\underline{t}, \cdot), \underline{t} \in \mathbb{T}\}$ with respect to scalar product $\langle \cdot, \cdot \rangle_{L^2_\gamma(\mathbb{T})}$, since

$$\left\langle 1_{[0,t^{(1)}]\times\cdots\times[0,t^{(d)}]}, 1_{[0,s^{(1)}]\times\cdots\times[0,s^{(d)}]} \right\rangle_{\mathcal{H}_Y} = \langle K(\underline{t}, \cdot), K(\underline{s}, \cdot)\rangle_{L^2_\gamma(\mathbb{T})}.$$

The kernel K defines an operator $\mathcal{K}_\gamma : L^2_\gamma(\mathbb{T}) \to \mathcal{K}_\gamma(L^2_\gamma(\mathbb{T}))$ by

$$(\mathcal{K}_\gamma h)(\underline{t}) = \int_{\mathbb{T}^2} K(\underline{t}, \underline{u})h(\underline{v})\gamma(d\underline{u}, d\underline{v}).$$

Then

$$|(\mathcal{K}_\gamma h)(\underline{t}) - (\mathcal{K}_\gamma h)(\underline{s})| = \left|\langle K(\underline{t},\cdot) - K(\underline{s},\cdot), h\rangle_{L^2_\gamma}\right| \leq \|K(\underline{t},\cdot) - K(\underline{s},\cdot)\|_{L^2_\gamma}\|h\|_{L^2_\gamma}$$
$$= \left[\mathbb{E}(Y_{\underline{t}} - Y_{\underline{s}})^2\right]^{\frac{1}{2}} \|h\|_{L^2_\gamma},$$

and, thus, the function $\mathcal{K}_\gamma h$ is continuous and vanishes at $\underline{0} = (0,\ldots,0)'$.
For $n \in \mathbb{N}$, let $\mathcal{E}^n$ denote the class of step functions on $\mathbb{T}^n$. Let $L^2_R(\mathbb{T}^n)$ be defined in the same way as the space $L^2_\gamma(\mathbb{T}^n)$ but with R instead of γ. For $\varphi \in L^2_R(\mathbb{T}^n)$, let

$$\begin{aligned}(\mathcal{K}^*_{i,j}\varphi)(\underline{t}_1,...,\underline{t}_n) &:= \varphi(\underline{t}_1,...,\underline{t}_n)K_j(T_j,t_i^{(j)}) \\ &+ \int_{t_i^{(j)}}^{T_j} \left[\varphi(\underline{t}_1,...,\underline{t}_{i-1},\ell_j(\underline{t}_i,u),\underline{t}_{i+1},...,\underline{t}_n) - \varphi(\underline{t}_1,...,\underline{t}_n)\right] K_j(du,t_i^{(j)}),\end{aligned} \tag{4}$$

where $i = 1,\ldots,n$, $j = 1,\ldots,d$, and $\underline{t}_i = (t_i^{(1)},\ldots,t_i^{(d)})$, and

$$\ell_j(\underline{t}_i,u) := (t_i^{(1)},\ldots,t_i^{(j-1)},u,t_i^{(j+1)},\ldots,t_i^{(d)}).$$

First, let us consider the case $n = 1$ and define a linear operator $\Gamma^{(1)}_{K_1,\ldots,K_d} : \mathcal{E}^1 \to L^2_\gamma(\mathbb{T})$ by

$$(\Gamma^{(1)}_{K_1,\ldots,K_d}\varphi) := \mathcal{K}^*_{1,d}(\mathcal{K}^*_{1,d-1}(\ldots(\mathcal{K}^*_{1,1}\varphi)\ldots)), \tag{5}$$

For example, for $d = 2$, $\forall t = (t^{(1)},t^{(2)}) \in [0,T_1]\times[0,T_2]$,

$$\begin{aligned}(\Gamma^{(1)}_{K_1,K_2}\varphi)(t^{(1)},t^{(2)}) &= \varphi(t^{(1)},t^{(2)})K_1(T_1,t^{(1)})K_2(T_2,t^{(2)}) \\ &+ K_1(T_1,t^{(1)})\int_{t^{(2)}}^{T_2}\left[\varphi(t^{(1)},v) - \varphi(t^{(1)},t^{(2)})\right]K_2(dv,t^{(2)}) \\ &+ K_2(T_2,t^{(2)})\int_{t^{(1)}}^{T_1}\left[\varphi(u,t^{(2)}) - \varphi(t^{(1)},t^{(2)})\right]K_1(du,t^{(1)}) \\ &+ \int_{t^{(1)}}^{T_1}\int_{t^{(2)}}^{T_2} V\left(\varphi;\left(t^{(1)},u\right]\times\left(t^{(2)},v\right]\right)K_2(dv,t^{(2)})K_1(du,t^{(1)}),\end{aligned}$$

where $V(f;A)$ denotes the Vitali variation of the function f over the set A. Specifically, for $d = 2$, we have

$$V\left(\varphi;\left(t^{(1)},u\right]\times\left(t^{(2)},v\right]\right) = \varphi(u,v) - \varphi(t^{(1)},v) - \varphi(u,t^{(2)}) + \varphi(t^{(1)},t^{(2)}).$$

Moreover, note that for arbitrary $d \in \mathbb{N}$ and for all $\underline{t} = (t^{(1)},\ldots,t^{(d)}) \in \mathbb{T}$,

$$\left(\Gamma^{(1)}_{K_1,\ldots,K_d}1_{\left(0,b^{(1)}\right]\times\cdots\times\left(0,b^{(d)}\right]}\right)(\underline{t}) = \prod_{j=1}^{d} K_j\left(b^{(j)},t^{(j)}\right). \tag{6}$$

Lemma 1 *For all $\varphi \in \mathcal{E}^1$ and $h \in L^2_\gamma(\mathbb{T})$,*

$$\int_{\mathbb{T}\times\mathbb{T}} (\Gamma^{(1)}_{K_1,\ldots,K_d}\varphi)(\underline{u})h(\underline{v})\gamma(d\underline{u}, d\underline{v}) = \int_{\mathbb{T}} \varphi(\underline{t})(\mathcal{K}_\gamma h)(d\underline{t}). \tag{7}$$

Proof: Note that for all $h \in L^2_\gamma(\mathbb{T})$,

$$\int_{\mathbb{T}\times\mathbb{T}} (\Gamma^{(1)}_{K_1,\ldots,K_d} 1_{(0,b^{(1)}]\times\cdots\times(0,b^{(d)}]})(\underline{u})h(\underline{v})\gamma(d\underline{u}, d\underline{v}) = \int_{\mathbb{T}\times\mathbb{T}} \prod_{j=1}^{d} K_j(b^{(j)}, u^{(j)})h(\underline{v})\gamma(d\underline{u}, d\underline{v})$$

$$= \int_{\mathbb{T}\times\mathbb{T}} K(\underline{b}, \underline{u})h(\underline{v})\gamma(d\underline{u}, d\underline{v}) = (\mathcal{K}_\gamma h)(\underline{b}) = \int_{\mathbb{T}} 1_{(0,b^{(1)}]\times\cdots\times(0,b^{(d)}]}(\underline{t})(\mathcal{K}_\gamma h)(d\underline{t}).$$

Since every function $\varphi \in \mathcal{E}^1$ is a linear combination of indicator functions of the form $1_{(0,b^{(1)}]\times\cdots\times(0,b^{(d)}]}(\cdot)$ and the operators involved in (7) are linear, the required conclusion follows for arbitrary $\varphi \in \mathcal{E}^1$. □

Lemma 2 *For all $\varphi, \psi \in \mathcal{E}^1$,*

$$\int_{\mathbb{T}\times\mathbb{T}} (\Gamma^{(1)}_{K_1,\ldots,K_d}\varphi)(\underline{u})(\Gamma^{(1)}_{K_1,\ldots,K_d}\psi)(\underline{v})\gamma(d\underline{u}, d\underline{v}) = \int_{\mathbb{T}\times\mathbb{T}} \varphi(\underline{t})\psi(\underline{s})R(d\underline{t}, d\underline{s}). \tag{8}$$

Proof: For arbitrary $\underline{b} = (b^{(1)}, \ldots, b^{(d)})' \in \mathbb{T}$, let $[\underline{0}, \underline{b}]$ denote the rectangle $[0, b^{(1)}] \times \cdots \times [0, b^{(d)}]$. Take $\psi = 1_{[\underline{0},\underline{b}]}$. By Lemma 1, for all $\varphi \in \mathcal{E}^1$,

$$\int_{\mathbb{T}\times\mathbb{T}} (\Gamma^{(1)}_{K_1,\ldots,K_d}\varphi)(\underline{u})(\Gamma^{(1)}_{K_1,\ldots,K_d}\psi)(\underline{v})\gamma(d\underline{u}, d\underline{v}) = \int_{\mathbb{T}} \varphi(\underline{t})\Big(\mathcal{K}_\gamma(\Gamma^{(1)}_{K_1,\ldots,K_d}\psi)\Big)(d\underline{t})$$

$$= \int_{\mathbb{T}} \varphi(\underline{t})\Big(\mathcal{K}_\gamma(\Gamma^{(1)}_{K_1,\ldots,K_d}1_{[\underline{0},\underline{b}]})\Big)(d\underline{t}) = \int_{\mathbb{T}} \varphi(\underline{t})R(d\underline{t}, \underline{b}) = \int_{\mathbb{T}\times\mathbb{T}} \varphi(\underline{t})\psi(\underline{s})R(d\underline{t}, d\underline{s}).$$

The required conclusion then follows for arbitrary $\psi \in \mathcal{E}^1$ by linearity of the operators. □

Remark. In view of the lemmas 1 and 2 we may consider the operator $\Gamma^{(1)}_{K_1,\ldots,K_d} : L^2_R(\mathbb{T}) \to L^2_\gamma(\mathbb{T})$, determined by (5), such that for all $f, g \in L^2_R(\mathbb{T})$ satisfying

$$\int_{\mathbb{T}\times\mathbb{T}} |f(\underline{u})g(\underline{v})||R|(d\underline{u}, d\underline{v}) < \infty,$$

it follows that $\Gamma^{(1)}_{K_1,\ldots,K_d}f$ and $\Gamma^{(1)}_{K_1,\ldots,K_d}g$ belong to $L^2_\gamma(\mathbb{T})$, and

$$\begin{aligned}\langle f, g\rangle_{L^2_R(\mathbb{T})} &= \int_{\mathbb{T}\times\mathbb{T}} f(\underline{t})g(\underline{s})R(d\underline{t}, d\underline{s}) = \int_{\mathbb{T}\times\mathbb{T}} (\Gamma^{(1)}_{K_1,\ldots,K_d}f)(\underline{t})(\Gamma^{(1)}_{K_1,\ldots,K_d}g)(\underline{s})\gamma(d\underline{t}, d\underline{s}) \\ &= \Big\langle \Gamma^{(1)}_{K_1,\ldots,K_d}f, \Gamma^{(1)}_{K_1,\ldots,K_d}g \Big\rangle_{L^2_\gamma(\mathbb{T})}.\end{aligned} \tag{9}$$

Similarly, given arbitrary $n \in \mathbb{N}$, define a linear operator $\Gamma^{(n)}_{K_1,\ldots,K_d} : L^2_R(\mathbb{T}^n) \to L^2_\gamma(\mathbb{T}^n)$ by

$$(\Gamma^{(n)}_{K_1,\ldots,K_d}\varphi) := \mathcal{K}^*_{n,d}(\mathcal{K}^*_{n,d-1}(...(\mathcal{K}^*_{n,1}(\mathcal{K}^*_{n-1,d}(\mathcal{K}^*_{n-1,d-1}(...(\mathcal{K}^*_{n-1,1}(...\mathcal{K}^*_{1,d}(\mathcal{K}^*_{1,d-1}(...(\mathcal{K}^*_{1,1}\varphi)))...) \tag{10}$$

Then if $f, g \in L^2_R(\mathbb{T}^n)$ are such that the Lebesgue integral

$$\int_{\mathbb{T}^{2n}} |f(\underline{t}_1, \ldots, \underline{t}_n) g(\underline{s}_1, \ldots, \underline{s}_n)||R|(d\underline{t}_1, d\underline{s}_1) \ldots |R|(d\underline{t}_n, d\underline{s}_n) < \infty,$$

then $\langle f, g\rangle_{L^2_R(\mathbb{T}^n)} = \left\langle \Gamma^{(n)}_{K_1,\ldots,K_d} f, \Gamma^{(n)}_{K_1,\ldots,K_d} g \right\rangle_{L^2_\gamma(\mathbb{T}^n)}$.

Theorem 1 *Let X and Y be continuous d-parameter Gaussian random fields such that Y is a Volterra type random field with respect to X with kernel $K(\underline{t}, \underline{s}) = \prod_{j=1}^d K_j(t^{(j)}, s^{(j)})$, $\underline{t} = (t^{(1)}, \ldots, t^{(d)})$, $\underline{s} = (s^{(1)}, \ldots, s^{(d)}) \in \mathbb{T}$, satisfying the assumptions stated in the beginning of Section 2. Then multiple Itô-type stochastic integrals I^X_n and I^Y_n of order n with respect to random fields X and Y, respectively, are related via:*

$$I^Y_n(f) = I^X_n\left(\Gamma^{(n)}_{K_1,\ldots,K_d} f\right), \quad \forall f \in L^2_R(\mathbb{T}^n),$$

where $\Gamma^{(n)}_{K_1,\ldots,K_d}$ is given by (10) *(in terms of* (4)*).*

Proof: It suffices to prove the equality for $f = g^{\otimes n}$, $g \in L^2_R(\mathbb{T})$. Since

$$I^Y_1\left(1_{[0,t_1]\times\cdots\times[0,t_d]}\right) = Y_{t_1,\ldots,t_d} = \int_{\mathbb{T}} \left(\Gamma^{(1)}_{K_1,\ldots,K_d} 1_{[0,t_1]\times\cdots\times[0,t_d]}\right)(\underline{s}) dX_{\underline{s}}$$

and the step functions are dense in $L^2_R(\mathbb{T})$, it follows that $I^Y_1(g) = I^X_1\left(\Gamma^{(1)}_{K_1,\ldots,K_d} g\right)$ for all $g \in L^2_R(\mathbb{T})$. Therefore, by properties of the multiple Itô integral (see e.g. [1]) and isometry (9), for all $g \in L^2_R(\mathbb{T})$,

$$\begin{aligned}
I^Y_n\left(g^{\otimes n}\right) &= \|g\|^n_{L^2_R(\mathbb{T})} H_n\left(I^Y_1(g)/\|g\|_{L^2_R}\right) \\
&= \left\|\Gamma^{(1)}_{K_1,\ldots,K_d} g\right\|^n_{L^2_\gamma} H_n\left(I^X_1(\Gamma^{(1)}_{K_1,\ldots,K_d} g)/\|\Gamma^{(1)}_{K_1,\ldots,K_d} g\|_{L^2_\gamma})\right) \\
&= I^X_n\left((\Gamma^{(1)}_{K_1,\ldots,K_d} g)^{\otimes n}\right),
\end{aligned}$$

where $H_n(\cdot)$ denotes the nth Hermite polynomial ($H_0(x) = 1,\ H_1(x) = x,\ H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$ for all $n \geq 2$). $\square$

Corollary 1 *For all $f \in L^2_{R,s}(\mathbb{T}^n)$ and $g \in L^2_{\gamma,s}(\mathbb{T}^m)$,*

$$I^Y_n(f) I^X_m(g) = \sum_{k=0}^{m\wedge n} k! \binom{n}{k}\binom{m}{k} I^X_{n+m-2k}\left((\Gamma^{(n)}_{K_1,\ldots,K_d} f) \otimes_{\gamma,k} g\right),$$

$$\left((\Gamma^{(n)}_{K_1,\dots,K_d}f)\otimes_{\gamma,k} g\right)(\underline{t}_1,\dots,\underline{t}_{n+m-2k}) = \int_{\mathbb{T}^{2k}} (\Gamma^{(n)}_{K_1,\dots,K_d}f)(\underline{t}_1,\dots,\underline{t}_{n-k},\underline{s}_1,\dots,\underline{s}_k)$$
$$\times g(\underline{s}'_1,\dots,\underline{s}'_k,\underline{t}_{n-k+1},\dots,\underline{t}_{n+m-2k})\gamma(d\underline{s}_1,d\underline{s}'_1)\dots\gamma(d\underline{s}_k,d\underline{s}'_k)$$

Proof: The required result follows from the product formula for $I_n^X(h)I_m^X(g)$, where X is a Gaussian random field with covariance γ, and the transfer principle given in Theorem 1. □

Corollary 2 *For arbitrary $f \in L^2_{R,s}(\mathbb{T}^n)$ and $g \in L^2_{\gamma,s}(\mathbb{T}^m)$, $n,m \in \mathbb{N}$, the following statements are equivalent:*
(i) $I_n^Y(f)$ and $I_m^X(g)$ are independent;
(ii)
$$\int_{\mathbb{T}\times\mathbb{T}} g(\underline{t}_1,\dots,\underline{t}_{m-1},\underline{u})(\Gamma^{(n)}_{K_1,\dots,K_d}f)(\underline{s}_1,\dots,\underline{s}_{n-1},\underline{v})\gamma(d\underline{u},d\underline{v}) = 0 \ \ a.e.$$
(iii) $\mathrm{Cov}([I_n^Y(f)]^2,[I_m^X(g)]^2) = 0.$

Proof: The required result follows from Theorem 1 and the independence criterion established in [2]. □

Corollary 3 *Let $Y_{t^{(1)},\dots,t^{(d)}} = \int_0^{T_1}\cdots\int_0^{T_d}\prod_{j=1}^{d} K_j(t^{(j)},s^{(j)})dB^{H_1,\dots,H_d}_{s^{(1)},\dots,s^{(d)}}$, $\underline{t} \in \mathbb{T} = [0,T_1]\times \dots \times [0,T_d]$, be a Volterra-type d-parameter ($d \in \mathbb{N}$) continuous random field with respect to fractional Brownian sheet $B^{H_1,\dots,H_d}$ with Hurst multiindex $(H_1,\dots,H_d) \in (\frac{1}{2},1)^d$. Assume as in Theorem 1 that the kernel $K(\underline{t},\underline{s}) = \prod_{j=1}^d K_j(t^{(j)},s^{(j)})$ is of Volterra type. Let $f \in L^2_R(\mathbb{T}^n)$ be symmetric and not zero , where*

$$R(\underline{t},\underline{s}) = \prod_{j=1}^{d}\left(H_j(2H_j-1)\int_0^{T_j}\int_0^{T_j} K_j(t^{(j)},u^{(j)})K_j(s^{(j)},v^{(j)})\left|u^{(j)}-v^{(j)}\right|^{2H_j-2}du^{(j)}dv^{(j)}\right).$$

Then there exists a constant $C = C(n,f)$ such that for any sequence of symmetric functions $f_k \in L^2_R(\mathbb{T}^n)$ converging to f:

$$\left\|\mathcal{L}(I_n^Y(f) - \mathcal{L}(I_n^Y(f_k))\right\| \le C\left\|f-f_k\right\|^{1/n}_{L^2_R(\mathbb{T}^n)},$$

where $\mathcal{L}(X)$ denotes the law of X and $\|\mu\|$ is the total variation of a signed measure μ.

Proof: The result follows from the convergence in variation result of Davydov and Martynova [3], developed for multiple Itô integrals defined with respect to an orthogonal Gaussian measure, subsequent application of a stochastic transfer principle of Perez-Abreu and Tudor [4] for multiple fractional Itô-type integrals with respect to a fractional Brownian motion, where the latter principle is extended to a d-parameter case ($d > 1$), and an application of Theorem 1 together with the fact that

$$\|f-f_k\|_{L^2_R(\mathbb{T}^n)} = \left\|\Gamma^{(n)}_{K_1,\dots,K_d}f - \Gamma^{(n)}_{K_1,\dots,K_d}f_k\right\|_{L^2_\gamma(\mathbb{T}^n)},$$

where $\gamma(\underline{t}, \underline{s}) = \frac{1}{2^d} \prod_{j=1}^{d} ((t^{(j)})^{2H_j} + (s^{(j)})^{2H_j} - |t^{(j)} - s^{(j)}|^{2H_j})$. $\square$

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