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# Vortex theory approach to stochastic hydrodynamics

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**Abstract**

The objective of the paper is to study a jump-diffusion type vorticity model, describing evolution of an incompressible homogeneous viscous fluid in $\mathbb{R}^2$ in terms of its rotation. The model arises from a particle systems perspective, adopted in the point vortex theory, and represents a measure-valued stochastic partial differential equation (SPDE) whose solution, under certain conditions, is an empirical process generated by a finite system of randomly moving vortices, which interact via a (regularized) logarithmic potential and are driven by suitable independent space-time Wiener processes and compensated Poisson random measure. A continuous diffusion approximation to the above vorticity model is also presented.
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## 1. Introduction

Turbulence has long intrigued physicists and mathematicians alike. While being a widely observed and an extremely important phenomenon for engineering and environmental applications related to atmospheric and oceanic sciences, it presents such an abundance of open mathematical problems and experimental and numerical challenges, which few (if any) other natural phenomena can really match. The mathematical history of fluid dynamics began with Euler, who in 1755 came up with a model, which in modern notation reads as:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + u(x,t)\cdot\nabla u(x,t) = -\nabla p(x,t), \\ \nabla\cdot u(x,t) = 0, \end{cases} \tag{1}$$

and goes under the name of Euler's equations of motion for an ideal fluid. Here $u(x,t)$ and $p(x,t)$ are, respectively, the fluid velocity and pressure at the spatial point $x$ at time $t$. One of the main ideas missing in that model was the necessity to account for viscous dissipation that is present due to the friction of one volume element of the fluid against neighboring ones. The appropriate term was added by Navier in 1827 and by Stokes in 1845, and, as a result, the celebrated "Navier–Stokes equations for an incompressible homogeneous viscous fluid" emerged, with

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\cdot\nabla u(x,t) = -\nabla p(x,t) + \nu\Delta u(x,t), \tag{2}$$

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replacing the first equation in (1), and $\nu$ being the material parameter which denotes the kinematic viscosity of the fluid. Since then, for almost two centuries now, generations of mathematicians have worked on a number of natural and fundamental questions related to the Navier–Stokes model. Surprisingly many of these questions remain open and represent some of the most interesting and challenging problems in modern mathematics. On the other hand, from both experimental and numerical perspectives, finding adequate descriptions of the dynamics of turbulent flows and their simulation and prediction has proved to be an equally interesting and difficult task. Solutions to the Navier–Stokes equations (NSE) are strongly affected by the nonlinearity and give rise to instabilities, while the actual flow one obtains in the laboratory is clearly a function of the experiment and is characterized as unsteady, chaotic and vortical.

Experimental observations therefore suggest that the chaotic nature of such hydrodynamical phenomena calls for their statistical or stochastic formulation. The goal of the present paper is to motivate and study the properties of a jump-diffusion type model for the vorticity of an incompressible homogeneous viscous fluid in $\mathbb{R}^2$. The latter model was introduced by Amirdjanova in [1] and represents a non-linear (signed) measure-valued stochastic partial differential equation of the form:

$$\begin{aligned} \mathrm{d}\langle\omega_t, f\rangle &= \langle\omega_t, \nu\Delta f + (U_\delta\omega_t)\cdot\nabla f\rangle\mathrm{d}t + \int_\Lambda \langle\omega_t, f(\cdot+\vartheta h(t,\cdot,\lambda)) - f - \vartheta h(t,\cdot,\lambda)\cdot\nabla f\rangle \Pi(\mathrm{d}\lambda)\mathrm{d}t \\ &\quad + \sqrt{2\nu}\sum_{j=1}^{2}\int_{\mathbb{R}^2}\langle\omega_t, \Gamma(\cdot,v)\partial_j f\rangle W_j(\mathrm{d}t\mathrm{d}v) + \int_\Lambda \langle\omega_{t-}, f(\cdot+\vartheta h(t,\cdot,\lambda)) - f\rangle\widetilde{\mathcal{N}}(\mathrm{d}t\mathrm{d}\lambda), \end{aligned} \tag{3}$$

for all $f \in C_b^2(\mathbb{R}^2)$, where $\omega$ describes the vorticity process of the fluid, $(U_\delta\omega_t)(x) = \int_{\mathbb{R}^2} K_\delta(x-r)\omega_t(\mathrm{d}r)$ represents a regularized measure-valued version of the Biot–Savart law (recall that in the deterministic case the classical Biot–Savart law recovers the velocity field from the vorticity process), and the martingales driving the equation are given by the two-dimensional Brownian sheet $(W_1, W_2)$ and a (compensated) Poisson random measure $\widetilde{\mathcal{N}}$. Note that when $\vartheta = \nu = 0$, the model effectively reduces to the incompressible 2-dim deterministic Euler equation (in its vorticity form). When $\theta = 0$ but $\nu > 0$, one recovers the stochastic vorticity model of Kotelenez [2].

The paper is organized as follows: Section 2 provides the mathematical description and motivation behind model (3), Section 3 presents results on several metric spaces needed later, Section 4 is devoted to the question of existence of solutions to (3) and construction of diffusion approximations. Finally, some concluding remarks are given in Section 5.

## 2. Stochastic vorticity model

To motivate our discussion of stochastic vorticity modeling, let us consider an ideal fluid whose evolution is governed by the Euler model (1) in the domain $D = \mathbb{R}^2$, supplied with the boundary condition $u(x,t) \to 0$ as $|x| \to \infty$ and initial condition $u(x,t)|_{t=0} = u^0(x)$. In view of the incompressibility condition, the pressure $p$ is determined by the velocity field via: $\Delta p = -\nabla\cdot((u\cdot\nabla)u)$, thus Euler's equation can be viewed as an evolution equation for the velocity field alone. The corresponding vorticity process $\omega_t(x) = \mathrm{curl}(u(x,t))$ describes rotation in the fluid and allows an equivalent formulation of the problem in terms of vorticity:

$$\frac{\partial\omega_t(x)}{\partial t} + \left(\left(\int_{\mathbb{R}^2} K(x-y)\omega_t(y)\mathrm{d}y\right)\cdot\nabla\right)\omega_t(x) = 0, \tag{4}$$

where, with a slight abuse of notation, $\omega$ stands for the only nonvanishing component of the vorticity field, i.e. $\omega_t(x) = \partial_{x_1}u_2(x,t) - \partial_{x_2}u_1(x,t)$, and kernel $K$, given by $K(x-y) = -\frac{1}{2\pi}\begin{pmatrix}\partial_{x^2}\\ -\partial_{x^1}\end{pmatrix}\ln|x-y| = -\frac{(x-y)^\perp}{2\pi|x-y|^2}$, is the Biot–Savart kernel that allows one to invert the relationship between velocity and vorticity to obtain:

$$u(x,t) = \int_{\mathbb{R}^2} K(x-y)\omega_t(y)\mathrm{d}y. \tag{5}$$

(For arbitrary $x = (x_1, x_2)'$ in $\mathbb{R}^2$, $x^\perp := (x_2, -x_1)'$). Next let us consider a (signed) measure-valued vorticity process $\tilde{\omega}_t(\mathrm{d}x) = \omega_t(x)\mathrm{d}x$ and look at the weak form of the equation (4), namely, for all sufficiently regular test functions $f$,

$$\mathrm{d}\langle\tilde{\omega}_t, f\rangle = \langle\tilde{\omega}_t, (U\tilde{\omega}_t)\cdot\nabla f\rangle\mathrm{d}t, \tag{6}$$

where, for any (signed) measure $\mu$ defined on the Borel sets of $\mathbb{R}^2$, $\langle \mu, f\rangle := \int_{\mathbb{R}^2} f(x)\mu(\mathrm{d}x)$ and $(U\mu)(x) := \int_{\mathbb{R}^2} K(x-y)\mu(\mathrm{d}y)$. Spatial discretization of $\tilde{\omega}$ in (6) gives rise to an important class of numerical methods for simulation of fluid flows, known as point vortex methods and vortex blob methods. Intuitively, consider an empirical vorticity process $\tilde{\omega}_t^N = \sum_{i=1}^N a_i \delta_{x^{(i)}(t)}$, generated by a system of interacting "particles" or vortices, whose positions $(x^{(1)}(t), \ldots, x^{(N)}(t))'$ satisfy the system:

$$\mathrm{d}x^{(i)}(t) = \left(\int_{\mathbb{R}^2} K_\delta(x^{(i)}(t) - y)\tilde{\omega}_t^N(\mathrm{d}y)\right)\mathrm{d}t \equiv \sum_{j=1}^N a_j K_\delta(x^{(i)}(t) - x^{(j)}(t))\mathrm{d}t, \quad i = 1, \ldots, N, \tag{7}$$

where $K_\delta$ coincides with the Biot–Savart kernel $K$ outside the sphere of (small) radius $\delta$ but $K_\delta(0) = 0$. (The $\delta$-regularization is necessary since $K$ blows up when two particles approach each other). Then $\tilde{\omega}_t^N$ solves the following equation

$$\mathrm{d}\langle \tilde{\omega}_t, f\rangle = \langle \tilde{\omega}_t, (U_\delta \tilde{\omega}_t)\cdot \nabla f\rangle \mathrm{d}t, \tag{8}$$

which is just a $\delta$-regularized version of (6). This suggests use of interacting particle (vortex) systems to obtain approximations to weak solutions of the 2-dim incompressible Euler's equation.

Next, turning to the viscous case, it is easy to show that the rigorous analogue of model (6) with viscosity $\nu > 0$ is given by:

$$\mathrm{d}\langle \omega, f\rangle = \langle \omega_t, \nu\Delta f + (U\omega_t)\cdot \nabla f\rangle \mathrm{d}t, \tag{9}$$

which is the Navier–Stokes equation in its vorticity form for a two-dimensional incompressible homogeneous viscous flow, where the equation is viewed in its weak form. Its $\delta$-regularized version is given by:

$$\mathrm{d}\langle \omega, f\rangle = \langle \omega_t, \nu\Delta f + (U_\delta\omega_t)\cdot \nabla f\rangle \mathrm{d}t, \tag{10}$$

where both equations (9) and (10) model evolution of a (signed) measure-valued process $\tilde{\omega}$, but we henceforth suppress the "tilde" on top of $\omega$ for notational convenience. Then a natural question is how to construct interacting point vortex systems that generate vorticity profiles, whose evolution properly reflects the presence of "$\nu\Delta f$" in the equation. Numerically, this idea was very successfully explored by Chorin in [3], who introduced the first random point vortex method to simulate viscous incompressible flows. His (time-discretized) random vortex method was based on viscous splitting under which the classical Navier–Stokes equation was split into Euler's equation and the heat equation. Then Euler's equation was simulated by the (deterministic) inviscid vortex method, while the heat equation was simulated by independent Gaussian random walks of the vortices. Later Marchioro and Pulvirenti in [4] considered a continuous-time random vortex method with Gaussian random walks replaced by independent Brownian motions and proved a corresponding mean field type result.

These results are very important from both conceptual and numerical perspectives. Note, however, that random point vortex methods are typically designed for numerical approximation of the solution to the classical (deterministic) Navier–Stokes equations rather than for the stochastic partial differential equation (SPDE) modeling of turbulent flows, with the model of Kotelenez [2] plus the vorticity SPDEs discussed here (and in [5]) being the only exceptions. Yet, actual experiments show that the classical deterministic Navier–Stokes model loses its predictive power for large Reynolds numbers. Randomness occurs as a result of sensitivity to initial conditions and to perturbations, from the interplay of large numbers of degrees of freedom; microscopic perturbations get amplified to macroscopic scales and the flow one obtains is characterized as unsteady and chaotic.

Therefore, it is interesting to consider SPDE modeling of vorticity from the perspective of random point vortex theory, where the evolution of vorticity in the flow is viewed as an evolution generated by a system of randomly moving point vortices (i.e. "particles" which carry concentrations of vorticity, while the rest of the flow is irrotational), and therefore analyze the resulting "mezoscopic" vorticity SPDE for an underlying microscopic model of randomly moving vortices. Let us start by introducing the system of point vortices associated to our stochastic model (3).

**Skorohod-Itô type model for the evolution of a system of point vortices:** Let $x^{(i)}(t) \in \mathbb{R}^2$ be the position of the $i$th vortex at time $t \in [0, \infty)$, $x^{(i)} = (x_1^{(i)}, x_2^{(i)})'$, $i = 1, \ldots, N$. Each vortex has an associated vorticity intensity $a_i \in \mathbb{R}$, $i = 1, \ldots, N$, such that $\sum_{a_i>0} a_i = a^+$, $\sum_{a_i<0} a_i = -a^-$. The magnitude of $a_i$ represents the amount of circulation, while the sign of $a_i$ represents the direction of rotation associated with the $i$th vortex. Let

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis with right continuous filtration. All the stochastic processes are assumed to live on $\Omega$, be $\mathcal{F}_t$-adapted and $(\mathrm{d}P \times \mathrm{d}t)$-measurable, where $\mathrm{d}t$ is the Lebesgue measure on $[0, \infty)$. Suppose that positions of vortices for $t > 0$ satisfy the following system of SDEs:

$$\mathrm{d}x^{(i)}(t) = u_{\delta,t}(x^{(i)}(t))\mathrm{d}t + \sqrt{2\nu}\int_{\mathbb{R}^2} \mathbf{\Gamma}(x^{(i)}(t), v)W(\mathrm{d}t\mathrm{d}v) + \vartheta\int_{\Lambda} h(t, x^{(i)}(t-), \lambda)\widetilde{\mathcal{N}}(\mathrm{d}t\mathrm{d}\lambda), \tag{11}$$

$$u_{\delta,t}(x) = \sum_{j=1}^{N} a_j K_\delta(x - x^{(j)}(t)), \quad \forall x \in \mathbb{R}^2, \tag{12}$$

with the initial conditions: $x^{(i)}(0) = \xi^{(i)}$, $i = 1, \ldots, N$, where:

(a) $W(t, v) = (W_1(t, v), W_2(t, v))'$, $W_1$ and $W_2$ are given independent Brownian sheets on $\mathbb{R}_+ \times \mathbb{R}^2$ with mean zero, and variance $t|A|$, where $A$ is a Borel set in $\mathbb{R}^2$ with finite Lebesgue measure $|A|$;

(b) $\mathcal{N}$ is a given Poisson random measure on $\mathbb{R}_+ \times \Lambda$, independent of $W$, with a characteristic measure $\Pi$ on $\Lambda$ (i.e. $\Pi(\mathrm{d}\lambda)$ is a $\sigma$-finite measure on a measurable space $(\Lambda, \mathcal{E})$); $\widetilde{\mathcal{N}}(t, B) = \mathcal{N}(t, B) - t\Pi(B)$ is the compensated Poisson measure;

(c) For $0 < \delta < 1$, $K_\delta(\cdot)$ is a $\delta$-regularized Biot–Savart kernel satisfying: $K_\delta(0) = 0$, $K_\delta(x) = (\nabla^\perp g_\delta)(|x|)$ for $x \neq 0$ with $\nabla^\perp := \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right)'$, where $g_\delta$ is at least a twice continuously differentiable approximation to the fundamental solution $g(|x|) := (-1/2\pi)\ln|x|$ of the Poisson equation in $\mathbb{R}^2$, such that $g_\delta(|x|) \equiv (-1/2\pi)\ln|x|$ for $\delta \leq |x| \leq 1/\delta$ and $g_\delta$ has bounded derivatives up to order 2, satisfying $|g'_\delta(s)| \leq |g'(s)|$ and $|g''_\delta(s)| \leq |g''(s)|$ for all $s > 0$; ($|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^2$).

(d) For all $x, v \in \mathbb{R}^2$, $\mathbf{\Gamma}(x, v) := \begin{pmatrix} \Gamma(x,v) & 0 \\ 0 & \Gamma(x,v) \end{pmatrix}$, where the correlation functions $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ are defined to be bounded Borel-measurable functions, symmetric in $x, v \in \mathbb{R}^2$, such that $\int \Gamma^2(x, v)\mathrm{d}v = 1$.

(e) $\vartheta$ is a nonnegative constant (whose value depends on the kinematic viscosity $\nu$); $h(t, x, \lambda) : \mathbb{R}_+ \times \mathbb{R}^2 \times \Lambda \longrightarrow \mathbb{R}^2$ is a measurable function, such that $\forall t \geq 0, \forall x \in \mathbb{R}^2$,

$$\int_{\Lambda} |h(t, x, \lambda)|^2 \Pi(\mathrm{d}\lambda) < \infty.$$

**Lemma 1.** *Let $\{\xi^{(i)}\}_{i=1}^N$ be $\mathbb{R}^2$-valued $\mathcal{F}_0$-adapted random variables, $\sum_{i=1}^N E|\xi^{(i)}|^2 < \infty$ and independent of $\sigma$-fields generated by $W$ and $\mathcal{N}$. Suppose there exist finite positive constants $D_1$ and $D_2$ such that the following conditions of Lipschitz and growth type are satisfied:*

$$\int_{\mathbb{R}^2} (\Gamma(x, v) - \Gamma(y, v))^2 \mathrm{d}v \leq D_1|x - y|^2, \quad \forall x, y \in \mathbb{R}^2, \tag{13}$$

$$\int_{\Lambda} |h(t, x, \lambda)|^2 \Pi(\mathrm{d}\lambda) \leq D_2, \quad \forall x \in \mathbb{R}^2, \forall t \geq 0, \tag{14}$$

$$\int_{\Lambda} |h(t, x, \lambda) - h(t, y, \lambda)|^2 \Pi(\mathrm{d}\lambda) \leq D_2|x - y|^2 \quad \textit{for } \forall x, y \in \mathbb{R}^2, \forall t \geq 0. \tag{15}$$

*Then the SDE system (11) and (12) with the initial condition $x^{(i)}(0) = \xi^{(i)}, \forall i = 1, \ldots, N$, has a unique strong solution.*

**Proof.** The solution is constructed by method of successive approximations. Consider the sequence $\{X_n(t)\}_{n=0}^\infty$, where $X_n(t) = (x_1^{(1),n}, x_2^{(1),n}, \ldots, x_1^{(N),n}, x_2^{(N),n})$ and $x^{(i),0}(t) = \xi^{(i)}$ a.s.,

$$x^{(i),n+1}(t) = \xi^{(i)} + \int_0^t u_{\delta,s}(x^{(i),n}(s))\mathrm{d}s + \sqrt{2\nu}\int_0^t\int_{\mathbb{R}^2} \mathbf{\Gamma}(x^{(i),n}(s), v)W(\mathrm{d}s\mathrm{d}v)$$
$$+ \vartheta\int_0^{t+}\int_{\Lambda} h(s, x^{(i),n}(s-), \lambda)\widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda), \quad i = 1, \ldots, N.$$

By Walsh [6], if $q_1(t)$, $q_2(t)$ are $\mathbb{R}^2$-valued adapted stochastic processes, then the stochastic integrals $\int_0^t \int_{\mathbb{R}^2} \mathbf{\Gamma}(q_i(s), v)W(\mathrm{d}s\mathrm{d}v)$, for $i = 1, 2$, are $\mathbb{R}^2$-valued square-integrable continuous martingales with the mutual quadratic variation given by:

$$\left\langle\left\langle \int_0^{\cdot} \int_{\mathbb{R}^2} \Gamma(q_1(s), v)W_i(\mathrm{d}s\mathrm{d}v), \int_0^{\cdot} \int_{\mathbb{R}^2} \Gamma(q_2(s), v)W_j(\mathrm{d}s\mathrm{d}v) \right\rangle\right\rangle_t$$
$$= \begin{cases} \int_0^t \int_{\mathbb{R}^2} \Gamma(q_1(s), v)\Gamma(q_2(s), v)\mathrm{d}s\mathrm{d}v, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

By application of Schwartz, Minkowskii's and Doob-Kolmogorov's inequalities and using the given Lipschitz and growth conditions, one can easily show that

$$E \sup_{0\le s\le t} \|X_{n+1}(s) - X_n(s)\|^2 \le c \int_0^t E\|X_n(s) - X_{n-1}(s)\|^2\mathrm{d}s$$
$$\le c^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} E\|X_1(s_n) - X_0(s_n)\|^2\mathrm{d}s_1 \cdots \mathrm{d}s_n$$

for some positive constant $c$ (which does not depend on $n$). Classical arguments then show that $\{X_n\}_{n=0}^{\infty}$ converges in $L_2(P)$ to a $2N$-dimensional $\mathcal{F}_t$-measurable cadlag process $X$, which is a solution to (11) and (12). Uniqueness of the strong solution follows by Gronwall's inequality. The argument is standard and we omit the details. □

**Lemma 2.** *Let $(\omega_t^N)_{t\ge 0}$ be the empirical signed measure-valued process*

$$\omega_t^N = \sum_{i=1}^N a_i \delta_{x^{(i)}(t)}, \tag{16}$$

*associated with the evolution of point vortices given in Lemma 1. Then $(\omega_t^N)$ is a strong solution to the SPDE (3) with the initial condition $\omega_t|_{t=0} = \sum_{i=1}^N a_i \delta_{\xi^{(i)}}$.*

**Proof.** The result follows by application of the generalized Itô formula and upon noting that for all $x \in \mathbb{R}^2$, $u_{\delta,t}(x) = \int_{\mathbb{R}^2} K_\delta(x - y)\omega_t^N(\mathrm{d}y) = (U_\delta\, \omega_t^N)(x)$. □

## 3. Wasserstein metric, its modifications and properties

In this section we discuss a variant of the Wasserstein distance for signed measures, which will be useful to us in the next section.

Consider $\mathbb{R}^2$ equipped with the uniformly bounded metric $d(x, y) = \min(|x - y|, 1)$, where $|x|^2 = (x_1)^2 + (x_2)^2$. For $b \in \mathbb{R}$, let $\mathcal{M}(b)$ be the space of all Borel finite signed measures $\mu$ on $\mathbb{R}^2$ with $\mu(\mathbb{R}^2) = b$. For given fixed $a^+, a^- \ge 0$ let $\mathcal{M}(a^+, a^-)$ be the space of all Borel signed measures $\mu$ on $\mathbb{R}^2$, such that $\mu^+(\mathbb{R}^2) = a^+$ and $\mu^-(\mathbb{R}^2) = a^-$, where $\mu^+$ and $\mu^-$ are the positive and negative parts of $\mu$ according to the Jordan decomposition of $\mu$. For $b \ge 0$ let $\mathcal{M}^+(b)$ be the space of all nonnegative Borel measures $\mu$ on $\mathbb{R}^2$ with $\mu(\mathbb{R}^2) = b$. Finally, $\forall \mu_1, \mu_2 \in \mathcal{M}^+(1)$, let $\mathcal{J}(\mu_1, \mu_2)$ be the set of all probability measures $Q$ on $\mathbb{R}^2 \times \mathbb{R}^2$ such that $Q(A \times \mathbb{R}^2) = \mu_1(A)$ and $Q(\mathbb{R}^2 \times B) = \mu_2(B)$ for arbitrary Borel sets $A, B \subset \mathbb{R}^2$.

For $\mu_1, \mu_2 \in \mathcal{M}^+(1)$, the Wasserstein distance is given by:

$$R_1(\mu_1, \mu_2) := \inf_{P\in\mathcal{J}(\mu_1,\mu_2)} \int d(x, y)P(\mathrm{d}x, \mathrm{d}y). \tag{17}$$

Moreover, for arbitrary finite signed measures $\mu_1, \mu_2$, define

$$V(\mu_1, \mu_2) := \sup_{\|\varphi\|_L\le 1} |\langle \mu_1 - \mu_2, \varphi\rangle|, \tag{18}$$

where $\forall \varphi \in C(\mathbb{R}^2, \mathbb{R})$, the pseudonorm $\|\cdot\|_L$ is given by:

$$\|\varphi\|_L := \sup_{x\neq y; x,y\in\mathbb{R}^2} \frac{|\varphi(x)-\varphi(y)|}{d(x,y)}. \tag{19}$$

Then, for any given $b$, $(\mathcal{M}(b), V)$ is a complete metric space. By the Kantorovich–Rubinstein theorem ([7,8]), the distances $R_1$ and $V$ coincide on $\mathcal{M}^+(1)$. Moreover, since $d(\cdot,\cdot)$ is uniformly bounded, by Dobrushin's theorem [9], the topology induced by $R_1$ on $\mathcal{M}^+(1)$ is equivalent to the usual weak convergence topology and makes it separable and complete. Moreover, by a result of De Acosta [8], the set $\{\mu \in \mathcal{M}^+(1) : \mu \text{ has finite support}\}$ is dense in $(\mathcal{M}^+(1), R_1)$.

The following modification of the Wasserstein metric $R_1$ for $\mathcal{M}(a^+, a^-)$ was suggested in [2]: $\forall \mu_1, \mu_2 \in \mathcal{M}(a^+, a^-)$ define $\gamma$ on $\mathcal{M}(a^+, a^-)$ by

$$\gamma(\mu_1,\mu_2) := \left[\inf_{\substack{P^+\in\mathcal{J}(\mu_1^+/a^+,\,\mu_2^+/a^+)\\ P^-\in\mathcal{J}(\mu_1^-/a^-,\,\mu_2^-/a^-)}} \int\int d^2(x,y)((a^+)^2P^+ + (a^-)^2P^-)(\mathrm{d}x\mathrm{d}y)\right]^{\frac{1}{2}}. \tag{20}$$

Note that if $\min(a^+, a^-) = 0$ and $a := a^+ + a^- > 0$, then $\gamma$ (by definition) is given by:

$$\gamma(\mu_1,\mu_2) := a\left[\inf_{P\in\mathcal{J}(|\mu_1|/a,\,|\mu_2|/a)} \int\int d^2(x,y)P(\mathrm{d}x\mathrm{d}y)\right]^{\frac{1}{2}}. \tag{21}$$

**Lemma 3.** *For any given $a^+, a^- \geq 0$ with $\max(a^+, a^-) > 0$, $(\mathcal{M}(a^+, a^-), \gamma)$ is a metric space. Moreover, the set $\mathcal{M}_p(a^+, a^-)$ of finite linear combinations of point measures from $\mathcal{M}(a^+, a^-)$ is dense in $(\mathcal{M}(a^+, a^-), \gamma)$.*

**Proof.** The proof is an extension of the argument in [8]. First let us show that $\gamma$ is a pseudometric on $\mathcal{M}(a^+, a^-)$. Clearly, $\forall \mu_1, \mu_2 \in \mathcal{M}(a^+, a^-)$, $\gamma(\mu_1, \mu_2) = \gamma(\mu_2, \mu_1) \geq 0$. To prove the triangle inequality, without loss of generality, assume that $a^+, a^-$ are both nonzero. Let $X, Y, Z$ be copies of $\mathbb{R}^2$. Fix an arbitrary small $\epsilon > 0$. Let $P^+ \in \mathcal{J}(\frac{\mu_1^+}{a^+}, \frac{\mu_2^+}{a^+})$ and $P^- \in \mathcal{J}(\frac{\mu_1^-}{a^-}, \frac{\mu_2^-}{a^-})$ be the probability measures on $X \times Y$, and $Q^+ \in \mathcal{J}(\frac{\mu_2^+}{a^+}, \frac{\mu_3^+}{a^+})$, $Q^- \in \mathcal{J}(\frac{\mu_2^-}{a^-}, \frac{\mu_3^-}{a^-})$ be the probability measures on $Y \times Z$, such that

$$\left[\int\int d^2(x,y)\left((a^+)^2P^+ + (a^-)^2P^-\right)(\mathrm{d}x\mathrm{d}y)\right]^{\frac{1}{2}} \leq \gamma(\mu_1,\mu_2) + \frac{\epsilon}{2}, \tag{22}$$

$$\left[\int\int d^2(y,z)\left((a^+)^2Q^+ + (a^-)^2Q^-\right)(\mathrm{d}y\mathrm{d}z)\right]^{\frac{1}{2}} \leq \gamma(\mu_2,\mu_3) + \frac{\epsilon}{2}. \tag{23}$$

Then, by Corollary A.2. in [8], there exist probability measures $S^+$, $S^-$ on $X \times Y \times Z$ such that for all Borel sets $A, B, C$ in $\mathbb{R}^2$, $S^+(A\times B\times\mathbb{R}^2) = P^+(A\times B)$, $S^+(\mathbb{R}^2\times B\times C) = Q^+(B\times C)$, and $S^-(A\times B\times\mathbb{R}^2) = P^-(A\times B)$, $S^-(\mathbb{R}^2\times B\times C) = Q^-(B\times C)$. In particular, it follows that $S^+(A\times\mathbb{R}^2\times\mathbb{R}^2) = \frac{\mu_1^+}{a^+}(A)$, $S^+(\mathbb{R}^2\times\mathbb{R}^2\times C) = \frac{\mu_3^+}{a^+}(C)$, and $S^-(A\times\mathbb{R}^2\times\mathbb{R}^2) = \frac{\mu_1^-}{a^-}(A)$, $S^-(\mathbb{R}^2\times\mathbb{R}^2\times C) = \frac{\mu_3^-}{a^-}(C)$. Therefore, by Minkowski's inequality,

$$\begin{aligned}
\gamma(\mu_1,\mu_3) &\leq \left[\iiint d^2(x,z)\left((a^+)^2S^+ + (a^-)^2S^-\right)(\mathrm{d}x\mathrm{d}y\mathrm{d}z)\right]^{\frac{1}{2}}\\
&\leq \left[\iiint d^2(x,y)\left((a^+)^2S^+ + (a^-)^2S^-\right)(\mathrm{d}x\mathrm{d}y\mathrm{d}z)\right]^{\frac{1}{2}}\\
&\quad + \left[\iiint d^2(y,z)\left((a^+)^2S^+ + (a^-)^2S^-\right)(\mathrm{d}x\mathrm{d}y\mathrm{d}z)\right]^{\frac{1}{2}}
\end{aligned}$$

$$= \left[\iint d^2(x,y)\left((a^+)^2P^+ + (a^-)^2P^-\right)(\mathrm{d}x\mathrm{d}y)\right]^{\frac{1}{2}} + \left[\iint d^2(y,z)\left((a^+)^2Q^+ + (a^-)^2Q^-\right)(\mathrm{d}y\mathrm{d}z)\right]^{\frac{1}{2}} \le \gamma(\mu_1,\mu_2) + \gamma(\mu_2,\mu_3) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the triangle inequality for $\gamma$ follows, and $\gamma$ is a pseudometric on $\mathcal{M}(a^+,a^-)$. Next let us show that $\gamma$ is a metric. Note that since $0 \le d(x,y) \le 1$ then $d^2(x,y) \le d(x,y)$ and $\gamma^2(\mu_1,\mu_2) \le (a^+)^2 R_1\left(\frac{\mu_1^+}{a^+},\frac{\mu_2^+}{a^+}\right) + (a^-)^2 R_1\left(\frac{\mu_1^-}{a^-},\frac{\mu_2^-}{a^-}\right)$. On the other hand, by the Cauchy–Bunyakovsky–Schwartz inequality,

$$\gamma^2(\mu_1,\mu_2) \ge \frac{1}{(a^+)^2+(a^-)^2}\left[(a^+)^2 R_1\left(\frac{\mu_1^+}{a^+},\frac{\mu_2^+}{a^+}\right) + (a^-)^2 R_1\left(\frac{\mu_1^-}{a^-},\frac{\mu_2^-}{a^-}\right)\right]^2.$$

Thus, $\gamma(\mu_1,\mu_2) = 0$ if and only if $R_1(\frac{\mu_1^+}{a^+},\frac{\mu_2^+}{a^+}) = R_1(\frac{\mu_1^-}{a^-},\frac{\mu_2^-}{a^-}) = 0$, which is true if and only if $\mu_1 = \mu_2$. Therefore, $\gamma$ is a metric.

To show the second part of the lemma, denote the set of all measures $\nu \in \mathcal{M}(a^+,a^-)$ such that $\nu$ is a finite linear combination of point measures by $\mathcal{M}_p(a^+,a^-)$, i.e. $\nu \in \mathcal{M}_p(a^+,a^-)$ if and only if $\exists\{c_i\}_{i=1}^k \in \mathbb{R}\setminus\{0\}$ and $\exists\{x_i\}_{i=1}^k \in \mathbb{R}^2$ for some finite $k \ge 2$ such that $\sum_{c_i>0} c_i = a^+$, $\sum_{c_i<0} c_i = -a^-$ and $\nu = \sum_{i=1}^k c_i\delta_{x_i}$.

Fix an arbitrary $\mu = \mu^+ - \mu^- \in \mathcal{M}(a^+,a^-)$. Denote by $B^+$ the support of $\mu^+$, and put $B^- := \mathbb{R}^2 \setminus B^+$. Without loss of generality, assume that $a^+ > 0$, $a^- > 0$. Fix arbitrary $\epsilon > 0$ and some $y_0^+ \in B^+$, $y_0^- \in B^-$. Consider a countable sequence of disjoint Borel sets

$$\{A_j^\pm\}_{j=1}^\infty := \left\{x = (x_1,x_2) \in B^\pm : \frac{k\epsilon}{2} < x_1 \le \frac{(k+1)\epsilon}{2}, \frac{l\epsilon}{2} < x_2 \le \frac{(l+1)\epsilon}{2}\right\}_{k,l\in\mathbf{Z}}.$$

Then $\bigcup_{j=1}^\infty A_j^+ = B^+$, $\bigcup_{j=1}^\infty A_j^- = B^-$ and $\mathrm{diam}(A_j^\pm) < \epsilon\ \forall j = 1,2,\ldots$. Since $\{A_j^\pm\}_{j=1}^\infty$ are disjoint,

$$\sum_{j=1}^\infty \int_{A_j^+} d^2(x,y_0^+)|\mu|(\mathrm{d}x) = \int_{\bigcup_{j=1}^\infty A_j^+} d^2(x,y_0^+)|\mu|(\mathrm{d}x) = \int_{B^+} d^2(x,y_0^+)|\mu|(\mathrm{d}x) \le |\mu|(B^+) = a^+ \quad (\text{since } d(\cdot,\cdot)\le 1).$$

Thus, by the dominated convergence theorem, $\exists k = k(\epsilon)$ such that

$$\sum_{j=k+1}^\infty \int_{A_j^+} d^2(x,y_0^+)|\mu|(\mathrm{d}x) = \int_{B^+\setminus\left(\bigcup_{j=1}^k A_j^+\right)} d^2(x,y_0^+)|\mu|(\mathrm{d}x) \le \epsilon^2.$$

Similarly, one shows that $\exists m = m(\epsilon)$ such that

$$\sum_{j=m+1}^\infty \int_{A_j^-} d^2(x,y_0^-)|\mu|(\mathrm{d}x) = \int_{B^-\setminus\left(\bigcup_{j=1}^m A_j^-\right)} d^2(x,y_0^-)|\mu|(\mathrm{d}x) \le \epsilon^2.$$

Let $n = n(\epsilon) = \max(k,m)$. We have found disjoint sets $A_1^+,\ldots,A_n^+,A_1^-,\ldots,A_n^-$ in $\mathbb{R}^2$ such that $\mathrm{diam}(A_j^\pm) < \epsilon$ for all $j = 1,\ldots,n$ and

$$\sum_{j=n+1}^\infty \int_{A_j^\pm} d^2(x,y_0^\pm)|\mu|(\mathrm{d}x) \le \epsilon^2. \tag{24}$$

For $j = 1,\ldots,n$ choose $y_j^\pm \in A_j^\pm$ and define probability measures $Q^+$, $Q^-$ on $\mathbb{R}^2\times\mathbb{R}^2$ by

$$Q^+(C_1\times C_2) := \sum_{j=1}^n \frac{\mu^+}{a^+}(A_j^+\cap C_1)\delta_{y_j^+}(C_2) + \frac{\mu^+}{a^+}\left(\left(\bigcup_{j=n+1}^\infty A_j^+\right)\cap C_1\right)\delta_{y_0^+}(C_2), \tag{25}$$

$$Q^-(C_1 \times C_2) := \sum_{j=1}^{n} \frac{\mu^-}{a^-}(A_j^- \cap C_1)\delta_{y_j^-}(C_2) + \frac{\mu^-}{a^-}\left(\left(\bigcup_{j=n+1}^{\infty} A_j^-\right) \cap C_1\right)\delta_{y_0^-}(C_2) \tag{26}$$

for all Borel sets $C_1, C_2$ in $\mathbb{R}^2$. Note that $Q^+(C_1 \times \mathbb{R}^2) = \frac{\mu^+}{a^+}(C_1)$ and

$$Q^+(\mathbb{R}^2 \times C_2) = \sum_{j=1}^{n} \frac{\mu^+}{a^+}(A_j^+)\delta_{y_j^+}(C_2) + \frac{\mu^+}{a^+}\left(\bigcup_{j=n+1}^{\infty} A_j^+\right)\delta_{y_0^+}(C_2).$$

Let $\nu^+ := \sum_{j=1}^{n} \mu^+(A_j^+)\delta_{y_j^+} + \mu^+\left(\bigcup_{j=n+1}^{\infty} A_j^+\right)\delta_{y_0^+}$. Then $Q^+ \in \mathcal{J}\left(\frac{\mu^+}{a^+}, \frac{\nu^+}{a^+}\right)$. Similarly, $Q^- \in \mathcal{J}\left(\frac{\mu^-}{a^-}, \frac{\nu^-}{a^-}\right)$, where $\nu^- := \sum_{j=1}^{n} \mu^-(A_j^-)\delta_{y_j^-} + \mu^-\left(\bigcup_{j=n+1}^{\infty} A_j^-\right)\delta_{y_0^-}$. Then

$$\nu := \nu^+ - \nu^- \in \mathcal{M}_p(a^+, a^-), \tag{27}$$

since $\nu^+$ and $\nu^-$ have disjoint supports. Moreover,

$$\begin{aligned}
\gamma(\mu, \nu) &\le \left[\iint d^2(x, y)((a^+)^2 Q^+ + (a^-)^2 Q^-)(\mathrm{d}x\mathrm{d}y)\right]^{\frac{1}{2}} \\
&= \left[a^+ \sum_{j=1}^{n} \int_{A_j^+} \left(\int d^2(x, y)\delta_{y_j^+}(\mathrm{d}y)\right)\mu^+(\mathrm{d}x) + a^+ \int_{\bigcup_{j=n+1}^{\infty} A_j^+} \left(\int d^2(x, y)\delta_{y_0^+}(\mathrm{d}y)\right)\mu^+(\mathrm{d}x)\right. \\
&\quad \left. + a^- \sum_{j=1}^{n} \int_{A_j^-} \left(\int d^2(x, y)\delta_{y_j^-}(\mathrm{d}y)\right)\mu^-(\mathrm{d}x) + a^- \int_{\bigcup_{j=n+1}^{\infty} A_j^-} \left(\int d^2(x, y)\delta_{y_0^-}(\mathrm{d}y)\right)\mu^-(\mathrm{d}x)\right]^{\frac{1}{2}} \\
&= \left[\sum_{j=1}^{n} \int_{A_j^+} d^2(x, y_j^+)\left(a^+\mu^+(\mathrm{d}x)\right) + a^+ \sum_{j=n+1}^{\infty} \int_{A_j^+} d^2(x, y_0^+)\mu^+(\mathrm{d}x)\right. \\
&\quad \left. + \sum_{j=1}^{n} \int_{A_j^-} d^2(x, y_j^-)\left(a^-\mu^-(\mathrm{d}x)\right) + a^- \sum_{j=n+1}^{\infty} \int_{A_j^-} d^2(x, y_0^-)\mu^-(\mathrm{d}x)\right]^{\frac{1}{2}} \\
&\le [\epsilon^2((a^+)^2 + a^+ + (a^-)^2 + a^-)]^{\frac{1}{2}} \le \text{const} \cdot \epsilon.
\end{aligned}$$

Therefore, $\mathcal{M}_p(a^+, a^-)$ is dense in $(\mathcal{M}(a^+, a^-), \gamma)$. □

**Remark.** Some caution is required in generalizing results valid for measures to signed measures. In particular, note that unless $\min(a^+, a^-) = 0$, the space $(\mathcal{M}(a^+, a^-), \gamma)$ is not complete. For example, consider a sequence $\mu_n = 2\delta_{1/n} - \delta_{-1/n}$, $n \ge 1$. Then $\mu_n \in \mathcal{M}(2, 1)$ for all $n \ge 1$ and $V(\mu_n, \delta_0) \to 0$ as $n \to \infty$. It is easy to see that the latter implies that $\{\mu_n\}_{n\ge1}$ is a Cauchy sequence in $(\mathcal{M}(2, 1), \gamma)$, but $\delta_0 \notin \mathcal{M}(2, 1)$.

For arbitrary $x = (x_1^{(1)}(t), x_2^{(1)}(t), \ldots, x_1^{(N)}(t), x_2^{(N)}(t))_{t\in[0,T]} \in D([0, T], \mathbb{R}^{2N})$ and $y = (y_1^{(1)}(t), y_2^{(1)}(t), \ldots, y_1^{(N)}(t), y_2^{(N)}(t))_{t\in[0,T]} \in D([0, T], \mathbb{R}^{2N})$ define a metric

$$\rho_N(x, y) = \inf_{\theta\in\Theta}\left\{\sup_{0\le t\le T} \sum_{i=1}^{N} |x^{(i)}(t) - y^{(i)}(\theta(t))| + \beta(\theta)\right\}, \tag{28}$$

where $\Theta$ is the set of strictly increasing continuous maps $\theta$ from $[0, T]$ onto itself with $\theta(0) = 0, \theta(T) = T$ and such that

$$\beta(\theta) := \sup_{0\le s<t\le T}\left|\log\frac{\theta(s) - \theta(t)}{s - t}\right| < \infty.$$

For arbitrary $\mu = (\mu_t)_{t\in[0,T]}$, $\nu = (\nu_t)_{t\in[0,T]} \in D([0, T], \mathcal{M}(a^+, a^-))$, define

$$\rho_\gamma(\mu, \nu) = \inf_{\theta\in\Theta}\left\{\sup_{0\le t\le T} \gamma(\mu_t, \nu_{\theta(t)}) + \beta(\theta)\right\}. \tag{29}$$

**Lemma 4.** *$(D([0,T],\mathcal{M}(a^+,a^-)),\rho_\gamma)$ is a metric space, and, when $\min(a^+,a^-)=0$, the space is complete.*

**Proof.** The proof is analogous to the proofs found in [10] for the classical Skorohod space and is omitted. □

## 4. Existence of solutions and diffusion approximation

**Lemma 5.** *Assume the conditions of Lemma 2 and let $(\omega^N_{t,\xi})$ be the empirical signed measure-valued process, given by (16), which is a strong solution to the SPDE (3) with the initial condition $\omega_t|_{t=0}=\sum_{i=1}^N a_i\delta_{\xi^{(i)}}$. Similarly, consider $\nu^N_{t,\eta}=\sum_{i=1}^N b_i\delta_{y^{(i)}(t)}$, where $(y_1^{(1)},y_2^{(1)},\dots,y_1^{(N)},y_2^{(N)})'$ is the solution to the system (11) and (12) but with the initial condition $y^{(i)}(0)=\eta^{(i)}$, implying that $(\nu^N_{t,\eta})$ is a strong solution to (3) with the initial condition $\omega_t|_{t=0}=\sum_{i=1}^N b_i\delta_{\eta^{(i)}}$. Moreover, suppose that $\sum_{a_i>0}a_i=\sum_{b_i>0}b_i=a^+$ and $\sum_{a_i<0}a_i=\sum_{b_i<0}b_i=-a^-$. Then for any $T>0$ there is a constant $c=c(T)>0$ such that for all $N\geq 2$*

$$E\sup_{0\leq s\leq T}\gamma^2(\omega^N_{s,\xi},\nu^N_{s,\eta})\leq cE\gamma^2(\omega^N_{0,\xi},\nu^N_{0,\eta}), \tag{30}$$

*where $\gamma$ is the metric on $\mathcal{M}(a^+,a^-)$ defined by (20).*

**Proof.** First, assume that the initial positions of the point vortices are deterministic. For arbitrary $\alpha\in\mathbb{R}^2$, let $q(t;\alpha)$ be the strong solution to the equation:

$$\begin{aligned}q(t;\alpha)=\alpha&+\int_0^t\left(\int_{\mathbb{R}^2}K_\delta(q(s;\alpha)-x)\omega^N_{s,\xi}(\mathrm{d}x)\right)\mathrm{d}s+\sqrt{2\nu}\int_0^t\int_{\mathbb{R}^2}\Gamma(q(s;\alpha),v)W(\mathrm{d}s\mathrm{d}v)\\&+\vartheta\int_0^{t+}\int_\Lambda h(s,q(s-;\alpha),\lambda)\widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda),\quad t\geq 0.\end{aligned} \tag{31}$$

Similarly, for an arbitrary $\beta\in\mathbb{R}^2$, let $r(t;\beta)$ be the strong solution of the equation:

$$\begin{aligned}r(t;\beta)=\beta&+\int_0^t\left(\int K_\delta(r(s;\beta)-x)\nu^N_{s,\eta}(\mathrm{d}x)\right)\mathrm{d}s+\sqrt{2\nu}\int_0^t\int_{\mathbb{R}^2}\Gamma(r(s;\beta),v)W(\mathrm{d}s\mathrm{d}v)\\&+\vartheta\int_0^{t+}\int_\Lambda h(s,r(s-;\beta),\lambda)\widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda),\quad t\geq 0.\end{aligned} \tag{32}$$

Note that $\forall i\in\{1,\dots,N\}$,

$$q(t;\xi^{(i)})=x^{(i)}(t)\text{ a.s.}\quad\text{and}\quad r(t;\eta^{(i)})=y^{(i)}(t)\text{ a.s.} \tag{33}$$

Consider the difference

$$\begin{aligned}q(t;\alpha)-r(t;\beta)&=(\alpha-\beta)+\int_0^t\left(\int_{\mathbb{R}^2}K_\delta(q(s;\alpha)-x)\left[\omega^N_{s,\xi}(\mathrm{d}x)-\nu^N_{s,\eta}(\mathrm{d}x)\right]\right)\mathrm{d}s\\&\quad+\int_0^t\left(\int_{\mathbb{R}^2}(K_\delta(q(s;\alpha)-x)-K_\delta(r(s;\beta)-x))\,\nu^N_{s,\eta}(\mathrm{d}x)\right)\mathrm{d}s\\&\quad+\sqrt{2\nu}\int_0^t\int_{\mathbb{R}^2}(\Gamma(q(s;\alpha);v)-\Gamma(r(s;\beta),v))\,W(\mathrm{d}s\mathrm{d}v)\\&\quad+\vartheta\int_0^{t+}\int_\Lambda(h(s,q(s-;\alpha),\lambda)-h(s,r(s-;\beta),\lambda))\,\widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda)\\&\equiv(\alpha-\beta)+J_1(t)+J_2(t)+J_3(t)+J_4(t).\end{aligned} \tag{34}$$

Upon using the Cauchy–Bunyakovsky–Schwartz inequality, the Lipschitz conditions and the Doob-Kolmogorov inequality, one obtains that for $k=2,3,4$,

$$E\sup_{0\leq s\leq t}|J_k(s)|^2\leq c_2\int_0^t E|q(s;\alpha)-r(s;\beta)|^2\mathrm{d}s \tag{35}$$

for some constant $c_2 = c_2(T)$.

$\forall t \geq 0$, by the Jordan decomposition of measures,

$$\begin{aligned}
J_1(t) &\equiv \int_0^t \left( \int_{\mathbb{R}^2} K_\delta(q(s;\alpha) - x) \left[ \omega_{s,\xi}^N(\mathrm{d}x) - \nu_{s,\eta}^N(\mathrm{d}x) \right] \right) \mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}^2} K_\delta(q(s;\alpha) - x) \left[ \omega_{s,\xi}^{N,+}(\mathrm{d}x) - \nu_{s,\eta}^{N,+}(\mathrm{d}x) \right] \mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{R}^2} K_\delta(q(s;\alpha) - x) \left[ \omega_{s,\xi}^{N,-}(\mathrm{d}x) - \nu_{s,\eta}^{N,-}(\mathrm{d}x) \right] \mathrm{d}s \\
&= a^+ \int_0^t \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ K_\delta(q(s;\alpha) - x) - K_\delta(q(s;\alpha) - y) \right] Q_s^+(\mathrm{d}x\mathrm{d}y) \right) \mathrm{d}s \\
&\quad - a^- \int_0^t \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ K_\delta(q(s;\alpha) - x) - K_\delta(q(s;\alpha) - y) \right] Q_s^-(\mathrm{d}x\mathrm{d}y) \right) \mathrm{d}s,
\end{aligned}$$

for arbitrary $Q_t^+ \in \mathcal{J}\left( \frac{\omega_{t,\xi}^{N,+}}{a^+}, \frac{\nu_{t,\eta}^{N,+}}{a^+} \right)$, $Q_t^- \in \mathcal{J}\left( \frac{\omega_{t,\xi}^{N,-}}{a^-}, \frac{\nu_{t,\eta}^{N,-}}{a^-} \right)$.

Therefore, upon using the Cauchy–Bunyakovsky–Schwartz inequality and the Lipschitz condition on $K_\delta$ (with constant $D_3$), one obtains that

$$E \sup_{0 \leq s \leq t} |J_1(s)|^2 \leq 2 D_3 t E \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 \left( (a^+)^2 Q_s^+(\mathrm{d}x\mathrm{d}y) + (a^-)^2 Q_s^-(\mathrm{d}x\mathrm{d}y) \right) \mathrm{d}s \tag{36}$$

for arbitrary $Q_t^+ \in \mathcal{J}\left( \frac{\omega_{t,\xi}^{N,+}}{a^+}, \frac{\nu_{t,\eta}^{N,+}}{a^+} \right)$, $Q_t^- \in \mathcal{J}\left( \frac{\omega_{t,\xi}^{N,-}}{a^-}, \frac{\nu_{t,\eta}^{N,-}}{a^-} \right)$, for $\forall t \geq 0$.

Therefore, by arbitrariness of $Q_t^+$, $Q_t^-$, it follows that

$$\begin{aligned}
E \sup_{0 \leq s \leq t} |q(s;\alpha) - r(s;\beta)|^2 &\leq 5 |\alpha - \beta|^2 + c_2 \int_0^t E |q(s;\alpha) - r(s;\beta)|^2 \mathrm{d}s \\
&\quad + c_1 E \int_0^t \inf_{\substack{Q_s^+ \in \mathcal{J}_s^+ \\ Q_s^- \in \mathcal{J}_s^-}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x - y|^2 ((a^+)^2 Q_s^+(\mathrm{d}x\mathrm{d}y) + (a^-)^2 Q_s^-(\mathrm{d}x\mathrm{d}y)) \mathrm{d}s,
\end{aligned} \tag{37}$$

for suitable constants $c_1 = c_1(T)$ and $c_2 = c_2(T)$, where

$$\mathcal{J}_t^+ := \mathcal{J}\left( \frac{\omega_{t,\xi}^{N,+}}{a^+}, \frac{\nu_{t,\eta}^{N,+}}{a^+} \right), \quad \mathcal{J}_t^- := \mathcal{J}\left( \frac{\omega_{t,\xi}^{N,-}}{a^-}, \frac{\nu_{t,\eta}^{N,-}}{a^-} \right). \tag{38}$$

Or, expressed in terms of metric $\gamma$, we obtain that

$$E \sup_{0 \leq s \leq t} d^2(q(s;\alpha), r(s;\beta)) \leq 5 d^2(\alpha, \beta) + c_2 \int_0^t E d^2(q(s;\alpha), r(s;\beta)) \mathrm{d}s + c_1 \int_0^t E \gamma^2(\omega_{s,\xi}^N, \nu_{s,\eta}^N) \mathrm{d}s.$$

Then, $0 \leq t \leq T$, by the Gronwall's inequality, we have that

$$\begin{aligned}
E \sup_{0 \leq s \leq t} d^2(q(s;\alpha), r(s;\beta)) &\leq 5 d^2(\alpha, \beta) + c_1(T) \int_0^t E \gamma^2(\omega_{s,\xi}^N, \nu_{s,\eta}^N) \mathrm{d}s \\
&\quad + c_2(T) \int_0^t \mathrm{e}^{c_2(T)(t-s)} \left( 5 d^2(\alpha, \beta) + c_1(T) \int_0^s E \gamma^2(\omega_{\tau,\xi}^N, \nu_{\tau,\eta}^N) \mathrm{d}\tau \right) \mathrm{d}s \\
&\leq \left( 1 + c_2(T) \int_0^t \mathrm{e}^{c_2(T)(t-s)} \mathrm{d}s \right) \left( 5 d^2(\alpha, \beta) + c_1(T) \int_0^t E \gamma^2(\omega_{\tau,\xi}^N, \nu_{\tau,\eta}^N) \mathrm{d}\tau \right) \\
&= \mathrm{e}^{c_2(T)t} \left( 5 d^2(\alpha, \beta) + c_1(T) \int_0^t E \gamma^2(\omega_{\tau,\xi}^N, \nu_{\tau,\eta}^N) \mathrm{d}\tau \right).
\end{aligned}$$

I.e. we have that $\forall t \in [0, T]$

$$E \sup_{0\le s\le t} d^2(q(s;\alpha), r(s;\beta)) \le \mathrm{e}^{Tc_2(T)} \left(5d^2(\alpha,\beta) + c_1(T)\int_0^t E\gamma^2(\omega^N_{s,\xi}, \nu^N_{s,\eta})\mathrm{d}s\right). \tag{39}$$

For all bounded real continuous functions $F$ on $\mathbb{R}^4$ and arbitrary $Q^+ \in \mathcal{J}_0^+,\ Q^- \in \mathcal{J}_0^-$ define measures $\Lambda_t^+$ and $\Lambda_t^-$ for $\forall t \ge 0$ by

$$\int_{\mathbb{R}^4} F(x,y)\Lambda_t^+(\mathrm{d}x\mathrm{d}y) := \int_{\mathbb{R}^4} F(q(t;\alpha), r(t;\beta))Q^+(\mathrm{d}\alpha\mathrm{d}\beta), \tag{40}$$

$$\int_{\mathbb{R}^4} F(x,y)\Lambda_t^-(\mathrm{d}x\mathrm{d}y) := \int_{\mathbb{R}^4} F(q(t;\alpha), r(t;\beta))Q^-(\mathrm{d}\alpha\mathrm{d}\beta). \tag{41}$$

Let us show that $\forall t \ge 0\ \Lambda_t^+ \in \mathcal{J}_t^+$ and $\Lambda_t^- \in \mathcal{J}_t^-$. Indeed, for an arbitrary Borel set $B \subset \mathbb{R}^2$, $\forall t \ge 0$, $\Lambda_t^+(B\times\mathbb{R}^2) = \int\int I_{\{x\in B\}}\Lambda_t^+(\mathrm{d}x\mathrm{d}y) = \int\int I_{\{\alpha\in\mathbb{R}^2:\, q(t;\alpha)\in B\}}Q^+(\mathrm{d}\alpha\mathrm{d}\beta) = Q^+(\widehat{B}_t\times\mathbb{R}^2)$, where $\widehat{B}_t := \{\alpha\in\mathbb{R}^2 : q(t;\alpha)\in B\}$, $\forall t\ge 0$. Since $Q^+ \in \mathcal{J}_0^+ \equiv \mathcal{J}\left(\frac{\omega^{N,+}_{0,\xi}}{a^+}, \frac{\nu^{N,+}_{0,\eta}}{a^+}\right)$, then, by (33),

$$\begin{aligned} Q^+(\widehat{B}_t\times\mathbb{R}^2) &= \frac{\omega^{N,+}_{0,\xi}}{a^+}(\widehat{B}_t) = \frac{1}{a^+}\sum_{a_i>0} a_i\delta_{\xi^{(i)}}(\widehat{B}_t) = \frac{1}{a^+}\sum_{a_i>0} a_i\delta_{q(t;\xi^{(i)})}(B) \\ &= \frac{1}{a^+}\sum_{a_i>0} a_i\delta_{x^{(i)}(t)}(B) = \frac{\omega^{N,+}_{t,\xi}}{a^+}(B). \end{aligned}$$

Thus, we have shown that for an arbitrary Borel set $B\subset\mathbb{R}^2$, $\forall t\ge 0$,

$$\Lambda_t^+(B\times\mathbb{R}^2) = Q^+(\widehat{B}_t\times\mathbb{R}^2) = \frac{\omega^{N,+}_{t,\xi}}{a^+}(B).$$

Similarly, $\forall B\in\mathcal{B}(\mathbb{R}^2)$, $\forall t\ge 0$, the equality $\Lambda_t^+(\mathbb{R}^2\times B) = \frac{\nu^{N,\epsilon,+}_{t,\eta}}{a^+}(B)$ holds. Thus, $\forall t\ge 0$, $\Lambda_t^+\in\mathcal{J}_t^+$. The proof of $\Lambda_t^-\in\mathcal{J}_t^-\forall t\ge 0$ is analogous. Therefore,

$$\begin{aligned} E\sup_{0\le s\le t}\gamma^2(\omega^N_{s,\xi},\nu^N_{s,\eta}) &\equiv E\sup_{0\le s\le t}\inf_{\substack{P_s^+\in\mathcal{J}_s^+\\ P_s^-\in\mathcal{J}_s^-}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} d^2(x,y)((a^+)^2P_s^+ + (a^-)^2P_s^-)(\mathrm{d}x\mathrm{d}y) \\ &\le E\sup_{0\le s\le t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} d^2(x,y)((a^+)^2\Lambda_s^+ + (a^-)^2\Lambda_s^-)(\mathrm{d}x\mathrm{d}y) \\ &= E\sup_{0\le s\le t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} d^2(q(s;\alpha),r(s;\beta))((a^+)^2Q^+ + (a^-)^2Q^-)(\mathrm{d}\alpha\mathrm{d}\beta) \\ &\le \mathrm{e}^{Tc_2(T)}\int\int\left(5d^2(\alpha,\beta) + c_1(T)\int_0^t E\gamma^2(\omega^N_{s,\xi},\nu^N_{s,\eta})\mathrm{d}s\right)((a^+)^2Q^+ + (a^-)^2Q^-)(\mathrm{d}\alpha\mathrm{d}\beta) \\ &= c_3(T)\int\int d^2(\alpha,\beta)((a^+)^2Q^+ + (a^-)^2Q^-)(\mathrm{d}\alpha\mathrm{d}\beta) + c_4(T)\int_0^t E\gamma^2(\omega^N_{s,\xi},\nu^N_{s,\eta})\mathrm{d}s, \end{aligned}$$

where

$$c_3(T) := 5\mathrm{e}^{Tc_2(T)},\quad c_4(T) := ((a^+)^2 + (a^-)^2)c_1(T)\mathrm{e}^{Tc_2(T)}. \tag{42}$$

Then, by arbitrariness of $Q^+$, $Q^-$, it follows that $\forall t\in[0,T]$,

$$E\sup_{0\le s\le t}\gamma^2(\omega^N_{s,\xi},\nu^N_{s,\eta}) \le c_3(T)\gamma^2(\omega^N_{0,\xi},\nu^N_{0,\eta}) + c_4(T)\int_0^t E\gamma^2(\omega^N_{s,\xi},\nu^N_{s,\eta})\mathrm{d}s, \tag{43}$$

which, by Gronwall's inequality, implies that

$$E \sup_{0\le s\le t} \gamma^2(\omega^N_{s,\xi}, \nu^N_{s,\eta}) \le c_3(T)\gamma^2(\omega^N_{0,\xi}, \nu^N_{0,\eta}) \left(1 + c_4(T)\int_0^t \mathrm{e}^{c_4(T)(t-s)}\mathrm{d}s\right)$$
$$= c_3(T)\mathrm{e}^{tc_4(T)}\gamma^2(\omega^N_{0,\xi}, \nu^N_{0,\eta}).$$

Thus, $\forall T > 0$, there is a constant $c = c(T) := c_3(T)\mathrm{e}^{Tc_4(T)} > 0$ such that for all $N \in \mathbb{N}$

$$E \sup_{0\le s\le T} \gamma^2(\omega^N_{s,\xi}, \nu^N_{s,\eta}) \le c(T)\gamma^2(\omega^N_{0,\xi}, \nu^N_{0,\eta}). \tag{44}$$

The case of random (square-integrable, adapted) initial conditions follows similarly. □

**Theorem 1.** *For each $N \ge 2$ and arbitrary $\omega^N_0 \in L_2(\Omega, \mathcal{M}_p(a^+, a^-))$, let $(\omega^N_t)_{t\in[0,T]}$ be the strong solution of the stochastic vorticity equation (3) with the initial condition $\omega^N_0$, as given in Lemma 5. Suppose the conditions of Lemma 5 are satisfied and $\min(a^+, a^-) = 0$. Then the map $\pi$ from $L_2(\Omega, \mathcal{M}_p(a^+, a^-))$ to $L_2(\Omega, D([0, T], \mathcal{M}_p(a^+, a^-)))$, that maps*

$$\omega^N_0 \mapsto \omega^N = \left(\omega^N_t\right)_{t\in[0,T]} \quad \text{for every } \omega^N_0 \in L_2(\Omega; \mathcal{M}_p(a^+, a^-)),$$

*extends uniquely to a map $\widehat{\pi} : \omega_0 \mapsto \omega = (\omega_t)_{t\in[0,T]}$ from the space $L_2(\Omega, \mathcal{M}(a^+, a^-))$ to $L_2(\Omega, D([0, T], \mathcal{M}(a^+, a^-)))$. Moreover, for $\forall\omega_0, \nu_0 \in L_2(\Omega, \mathcal{M}(a^+, a^-))$, $\forall T > 0$, $\exists c = c(T) > 0$ (const) such that*

$$E\rho^2_\gamma(\widehat{\pi}(\omega_0), \widehat{\pi}(\nu_0)) \le cE\gamma^2(\omega_0, \nu_0). \tag{45}$$

**Proof.** By Lemma 5, the solution $(\omega^N_t)_{t\in[0,T]} \in L_2(\Omega, D([0, T]; \mathcal{M}_p(a^+, a^-)))$ to the stochastic vorticity equation (3) has a uniformly continuous dependence on its initial condition $\omega^N_0 \in L_2(\Omega, \mathcal{M}_p(a^+, a^-))$. Moreover, by Lemma 3, $\mathcal{M}_p(a^+, a^-)$ is dense in $(\mathcal{M}(a^+, a^-), \gamma)$, therefore one can extend the solution from discrete initial conditions to an arbitrary (adapted) initial condition $\omega_0 \in \mathcal{M}(a^+, a^-)$ by applying the following fact (see e.g. [11]):

*Let $X$ and $Y$ be metric spaces and $A$ be a dense subset of $X$ and let $Y$ be complete. If $f : A \longrightarrow Y$ is uniformly continuous then there exists a unique continuous map $F : X \longrightarrow Y$, with $F|_A = f$. $F$ is also uniformly continuous.*

Taking $X = \left(L_2(\Omega, \mathcal{M}(a^+, a^-)), (E\gamma^2(\cdot, \cdot))^{\frac{1}{2}}\right)$, $A = L_2(\Omega, \mathcal{M}_p(a^+, a^-))$, and the space $Y = \left(L_2(\Omega, D([0, T], \mathcal{M}(a^+, a^-))), (E\rho^2_\gamma(\cdot, \cdot))^{\frac{1}{2}}\right)$ (which is complete in view of Lemma 4) yields the desired result. The inequality follows at once from (30) upon taking the appropriate limits. □

**Theorem 2.** *Assume that the conditions of Theorem 1 hold and fix an arbitrary $\omega_0 \in L_2(\Omega, \mathcal{M}(a^+, a^-))$. Then $\widehat{\pi}(\omega_0) \in L_2(\Omega, D([0, T], \mathcal{M}(a^+, a^-)))$ is a solution to the stochastic vorticity equation (3) with the initial condition $\omega_0$.*

**Proof.** For an arbitrary $f \in C^m_b(\mathbb{R}^2, \mathbb{R})$, set

$$\|f\|_{b,m} := \max_{0\le|j|\le m} \sup_{x\in\mathbb{R}^2} \left|\frac{\partial^j}{(\partial x_1)^{j_1}(\partial x_2)^{j_2}} f(x)\right|,$$

where $|j| := j_1 + j_2$, $x = (x_1, x_2)'$. Note that, by the Kantorovich–Rubinstein theorem, $\forall\mu_1, \mu_2 \in \mathcal{M}(a^+, a^-)$,

$$\gamma^2(\mu_1, \mu_2) \ge \frac{1}{(a^+)^2 + (a^-)^2}\left[a^+ \sup_{\|\varphi\|_L\le 1} \left|\langle\mu^+_1 - \mu^+_2, \varphi\rangle\right| + a^- \sup_{\|\varphi\|_L\le 1} \left|\langle\mu^-_1 - \mu^-_2, \varphi\rangle\right|\right]^2$$
$$\ge c(a^+, a^-) \sup_{\|\varphi\|_L\le 1} |\langle\mu_1 - \mu_2, \varphi\rangle|^2$$

for some constant $c(a^+, a^-) > 0$.

Since $\mathcal{M}_p(a^+, a^-)$ is dense in $(\mathcal{M}(a^+, a^-), \gamma)$, for any given $\omega_0 \in L_2(\Omega, \mathcal{M}(a^+, a^-))$ we can choose a sequence $\{\omega_{0,k}\}_{k=1}^{\infty} \in L_2(\Omega, \mathcal{M}_p(a^+, a^-))$ such that

$$\gamma(\omega_{0,k}, \omega_0) \to 0 \text{ (a.s.) as } k \to \infty.$$

By Theorem 1, there exists a constant $0 < c(T) < \infty$ such that:

$$E\rho_\gamma^2(\widehat{\pi}(\omega_{0,k}), \widehat{\pi}(\omega_0)) \le c(T) E\gamma^2(\omega_{0,k}, \omega_0), \quad \forall k \ge 1. \tag{46}$$

For all $k \ge 1$, $\gamma^2(\omega_{0,k}, \omega_0) \le (a^+)^2 + (a^-)^2$ (a.s.) and $\gamma^2(\omega_{0,k}, \omega_0) \to 0$ (a.s.) as $k \to \infty$, thus $E\gamma^2(\omega_{0,k}, \omega_0) \to 0$ as $k \to \infty$, implying that

$$E\rho_\gamma^2(\widehat{\pi}(\omega_{0,k}), \widehat{\pi}(\omega_0)) = E \inf_{\theta \in \Theta} \left\{ \sup_{0 \le t \le T} \gamma(\omega_{t,k}, \omega_{\theta(t)}) + \beta(\theta) \right\}^2 \longrightarrow 0 \text{ as } k \to \infty, \tag{47}$$

where $(\omega_{t,k})_{t\in[0,T]} \equiv \widehat{\pi}(\omega_{0,k})$ and $(\omega_t)_{t\in[0,T]} \equiv \widehat{\pi}(\omega_0)$. Therefore, $\forall t \in [0, T]$,

$$\text{either } E\gamma^2(\omega_{t,k}, \omega_t) \longrightarrow 0 \text{ as } k \to \infty, \tag{48}$$

$$\text{or } E\gamma^2(\omega_{t,k}, \omega_{t-}) \longrightarrow 0 \text{ as } k \to \infty. \tag{49}$$

Moreover, processes from $D([0, T], \mathcal{M}(a^+, a^-))$ can have at most a countable number of jumps, while for all continuity points of $(\omega_t)$ the convergence of type (48) holds. Thus, by (46), for all but a countable number of points in $[0, T]$,

$$E \sup_{\|\varphi\|_L \le 1} \left\langle \omega_{t,k} - \omega_t, \varphi \right\rangle^2 \longrightarrow 0 \text{ as } k \to \infty. \tag{50}$$

To see that $(\omega_t)_{t\in[0,T]}$ is a solution to (3), let us look at the equation term by term.

By the Cauchy–Bunyakovsky–Schwartz inequality and Fubini's theorem,

$$E\left\{ \int_0^t \int_\Lambda \left\langle \omega_{s,k} - \omega_s, f(\cdot + \vartheta h(s, ., \lambda)) - f - \vartheta h(s, \cdot, \lambda) \cdot \nabla f \right\rangle \Pi(\mathrm{d}\lambda)\mathrm{d}s \right\}^2 \le t \int_0^t E\left\langle \omega_{s,k} - \omega_s, \psi_s \right\rangle^2 \mathrm{d}s,$$

where we set

$$\psi_s(x) := \int_\Lambda [f(x + \vartheta h(s, x, \lambda)) - f(x) - \vartheta h(s, x, \lambda) \cdot \nabla f(x)] \Pi(\mathrm{d}\lambda). \tag{51}$$

Suppose we prove that $\|\psi_s\|_L \le \text{const} < \infty \ \forall s \in [0, T]$, then the limit (50) and the dominated convergence theorem will imply: as $k \to \infty$,

$$\begin{aligned} &E\left\{ \int_0^t \int_\Lambda \left\langle \omega_{s,k} - \omega_s, f(\cdot + \vartheta h(s, \cdot, \lambda)) - f - \vartheta h(s, \cdot, \lambda) \cdot \nabla f \right\rangle \Pi(\mathrm{d}\lambda)\mathrm{d}s \right\}^2 \\ &\le t \int_0^t E\left\langle \omega_{s,k} - \omega_s, \psi_s \right\rangle^2 \mathrm{d}s \le T \ \text{const}^2 \int_0^t E\left\langle \omega_{s,k} - \omega_s, \frac{\psi_s}{\|\psi_s\|_L} \right\rangle^2 \mathrm{d}s \longrightarrow 0. \end{aligned} \tag{52}$$

Let us show that $\|\psi_s\|_L \le \text{const} < \infty, \forall s \in [0, T]$.

$$\|\psi_s\|_L = \sup_{x \ne z} \frac{|\psi_s(x) - \psi_s(z)|}{\min\{1, |x - z|\}} \le \sup_{|x-z| \ge 1} |\psi_s(x) - \psi_s(z)| + \sup_{|x-z| \le 1,\ x \ne z} \frac{|\psi_s(x) - \psi_s(z)|}{|x - z|}. \tag{53}$$

Consider the following Taylor's representation for $f \in C^m(\mathbb{R}^2, \mathbb{R})$:

$$f(x + y) = f(x) + \frac{(y \cdot \nabla) f(x)}{1!} + \cdots + \frac{(y \cdot \nabla)^{m-1} f(x)}{(m-1)!} + r_m, \tag{54}$$

with either an integral form of the remainder

$$r_m = \int_0^1 \frac{(1 - \kappa)^{m-1}}{(m-1)!} (y \cdot \nabla)^m f(x + \kappa y) \mathrm{d}\kappa, \tag{55}$$

or a differential form of the remainder

$$r_m = \frac{(y\cdot\nabla)^m f(x+\tau y)}{m!} \quad \text{for some } 0\le\tau\le 1. \tag{56}$$

Here, $y=(y_1,y_2)$, $(y\cdot\nabla)f = (y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2})f$,

$$(y\cdot\nabla)^2 f = \left(y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2}\right)^2 f = \left[(y_1)^2\frac{\partial^2}{(\partial x_1)^2}+2y_1y_2\frac{\partial^2}{\partial x_1\partial x_2}+(y_2)^2\frac{\partial^2}{(\partial x_2)^2}\right]f, \text{ etc.}$$

Note also that

$$\begin{aligned}\left|(y\cdot\nabla)^2 f\right| &\le \max_{|j_1+j_2|=2}\sup_{x\in\mathbb{R}^2}\left|\frac{\partial^2 f(x)}{(\partial x^1)^{j_1}(\partial x^2)^{j_2}}\right|\left((y^1)^2+2|y^1||y^2|+(y^2)^2\right)\\ &\le \|f\|_{b,3}\left(|y^1|+|y^2|\right)^2 \le 2\|f\|_{b,3}|y|^2.\end{aligned}$$

Then, on the one hand, for some $0\le\tau\le 1$, from (51) and the above inequality,

$$\begin{aligned}|\psi_s(x)| &\le \frac{\vartheta^2}{2!}\int_\Lambda |(h(s,x,\lambda)\cdot\nabla)^2 f|\,\big|_{x+\tau\vartheta h(s,x,\lambda)}\Pi(\mathrm{d}\lambda)\\ &\le \vartheta^2\|f\|_{b,3}\int_\Lambda |h(s,x,\lambda)|^2\,\Pi(\mathrm{d}\lambda) \le D_2\vartheta^2\|f\|_{b,3}<\infty,\end{aligned} \tag{57}$$

$\forall f\in C_b^3(\mathbb{R}^2,\mathbb{R})$. Since $D_2$ doesn't depend on $s$, then $\forall s\ge 0$,

$$\sup_{|x-z|\ge 1}|\psi_s(x)-\psi_s(z)| \le 2\sup_{x\in\mathbb{R}^2}|\psi_s(x)| \le 2D_2\vartheta^2\|f\|_{b,3}<\infty. \tag{58}$$

On the other hand, using the integral form of the remainder,

$$\psi_s(y) = \int_0^1\int_\Lambda \vartheta h(s,y,\lambda)\cdot\left(\nabla f(x)|_{x=y+\kappa\vartheta h(s,y,\lambda)}-\nabla f(x)|_{x=y}\right)\Pi(\mathrm{d}\lambda)\mathrm{d}\kappa, \tag{59}$$

which implies that

$$\begin{aligned}\psi_s(y)-\psi_s(z) &= \int_0^1\int_\Lambda \vartheta\left(h(s,y,\lambda)-h(s,z,\lambda)\right)\left(\nabla f(x)|_{x=y+\kappa\vartheta h(s,y,\lambda)}-\nabla f(x)|_{x=y}\right)\Pi(\mathrm{d}\lambda)\mathrm{d}\kappa\\ &\quad+\int_0^1\int_\Lambda \vartheta h(s,z,\lambda)\cdot\left(\nabla f(x)|_{x=y+\kappa\vartheta h(s,y,\lambda)}-\nabla f(x)|_{x=z+\kappa\vartheta h(s,z,\lambda)}\right)\Pi(\mathrm{d}\lambda)\mathrm{d}\kappa\\ &\quad+\int_0^1\int_\Lambda \vartheta h(s,z,\lambda)\cdot\left(\nabla f(x)|_{x=z}-\nabla f(x)|_{x=y}\right)\Pi(\mathrm{d}\lambda)\mathrm{d}\kappa.\end{aligned} \tag{60}$$

For notational convenience let $\alpha$ stand for the function:

$$\begin{aligned}&\alpha\equiv\alpha(s,y,z,\kappa,\rho,\lambda) := \rho\kappa\vartheta h(s,y,\lambda)+(1-\rho)\kappa\vartheta h(s,z,\lambda),\\ &\text{i.e. } z+\kappa\vartheta h(s,z,\lambda)+\rho(y-z+\kappa\vartheta[h(s,y,\lambda)-h(s,z,\lambda)]) = z+\rho(y-z)+\alpha,\end{aligned}$$

then, using the integral form of the remainder again, we write the equation (60) as:

$$\begin{aligned}&\psi_s(y)-\psi_s(z)\\ &= \vartheta^2\int_0^1\int_\Lambda \kappa(h(s,y,\lambda)-h(s,z,\lambda))\cdot\{(h(s,y,\lambda)\cdot\nabla)\nabla f(x)\}|_{x=y+\tau\kappa\vartheta h(s,y,\lambda)}\Pi(\mathrm{d}\lambda)\mathrm{d}\kappa\\ &\quad+\int_0^1\int_\Lambda \vartheta h(s,z,\lambda)\cdot\left(\int_0^1\{(y-z)\cdot\nabla\}(\nabla f(x)|_{x=z+\rho(y-z)+\alpha}-\nabla f(x)|_{x=z+\rho(y-z)})\mathrm{d}\rho\right)\Pi(\mathrm{d}\lambda)\mathrm{d}\kappa\\ &\quad+\vartheta^2\int_0^1\int_0^1\int_\Lambda \kappa h(s,z,\lambda)\cdot\{(h(s,y,\lambda)-h(s,z,\lambda))\cdot\nabla\}\nabla f(x)|_{x=z+\rho(y-z)+\alpha}\Pi(\mathrm{d}\lambda)\mathrm{d}\rho\mathrm{d}\kappa\\ &\equiv \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3,\end{aligned}$$

for some $0 \le \tau \le 1$, while the second term can be further written as:

$$\begin{aligned}\mathcal{T}_2 &= \int_0^1\int_0^1\int_\Lambda \vartheta h(s,z,\lambda)\cdot(\{\alpha\cdot\nabla\}\{(y-z)\cdot\nabla\}\nabla f(x)|_{x=z+\rho(y-z)+\tau'\alpha})\Pi(\mathrm{d}\lambda)\mathrm{d}\rho\mathrm{d}\kappa\\ &= \int_0^1\int_0^1\int_\Lambda \kappa\vartheta^2 h(s,z,\lambda)\cdot(\{[\rho h(s,y,\lambda)+(1-\rho)h(s,z,\lambda)]\cdot\nabla\}\{(y-z)\cdot\nabla\}\nabla f(x))|_{x=\widehat{x}}\Pi(\mathrm{d}\lambda)\mathrm{d}\rho\mathrm{d}\kappa\end{aligned}$$

for some $0 \le \tau' \le 1$, where $\widehat{x} := z+\rho(y-z)+\tau'\alpha(s,y,z,\kappa,\rho,\lambda)$.

Note that $\forall a=(a_1,a_2)',\ b=(b_1,b_2)',\ c=(c_1,c_2)'\in\mathbb{R}^2$,

$$b\cdot((c\cdot\nabla)\nabla f) = (b_1,b_2)(c_1\partial_1+c_2\partial_2)\begin{pmatrix}\partial_1 f\\ \partial_2 f\end{pmatrix} = (b_1,b_2)\begin{pmatrix}c_1\partial_1^2 f+c_2\partial_1\partial_2 f\\ c_1\partial_1\partial_2 f+c_2\partial_2^2 f\end{pmatrix},$$

$$|b\cdot((c\cdot\nabla)\nabla f)| \le \max_{|j_1+j_2|=2}\sup_{x\in\mathbb{R}^2}\left|\frac{\partial^2 f(x)}{(\partial x_1)^{j_1}(\partial x_2)^{j_2}}\right|(|b_1|+|b_2|)(|c_1|+|c_2|) \le 2|b||c|\|f\|_{b,3}.$$

Similarly,

$$b\cdot((a\cdot\nabla)(c\cdot\nabla)\nabla f) = b'\begin{pmatrix}a_1\partial_1^3+a_2\partial_1^2\partial_2 & a_1\partial_1^2\partial_2+a_2\partial_1\partial_2^2\\ a_1\partial_1^2\partial_2+a_2\partial_1\partial_2^2 & a_1\partial_1\partial_2^2+a_2\partial_2^3\end{pmatrix}c,$$

$$|b\cdot((a\cdot\nabla)(c\cdot\nabla)\nabla f)| \le 2\sqrt{2}|a||b||c|\max_{|j_1+j_2|=3}\sup_{x\in\mathbb{R}^2}\left|\frac{\partial^3 f(x)}{(\partial x_1)^{j_1}(\partial x_2)^{j_2}}\right| \le 2\sqrt{2}|a||b||c|\|f\|_{b,3}.$$

Then we get that

$$\begin{aligned}|\mathcal{T}_1| &\le 2\vartheta^2\|f\|_{b,3}\left(\int_0^1\kappa\mathrm{d}\kappa\right)\int_\Lambda |h(s,y,\lambda)-h(s,z,\lambda)|\,|h(s,y,\lambda)|\,\Pi(\mathrm{d}\lambda)\\ &\le \vartheta^2\|f\|_{b,3}\left(\int_\Lambda |h(s,y,\lambda)-h(s,z,\lambda)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac12}\left(\int_\Lambda |h(s,y,\lambda)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac12}\\ &\le D_2\vartheta^2\|f\|_{b,3}|y-z|.\end{aligned}$$

For the second term we have:

$$\begin{aligned}|\mathcal{T}_2| &\le 2\sqrt{2}\|f\|_{b,3}\vartheta^2\int_0^1\int_0^1\int_\Lambda \kappa\,|h(s,z,\lambda)|\,|\rho h(s,y,\lambda)+(1-\rho)h(s,z,\lambda)|\,|y-z|\,\Pi(\mathrm{d}\lambda)\mathrm{d}\rho\mathrm{d}\kappa\\ &\le \sqrt{2}\vartheta^2\|f\|_{b,3}|y-z|\int_0^1\int_\Lambda\left\{\rho|h(s,z,\lambda)|\,|h(s,y,\lambda)|+(1-\rho)|h(s,z,\lambda)|^2\right\}\Pi(\mathrm{d}\lambda)\mathrm{d}\rho\\ &\le \sqrt{2}D_2\vartheta^2\|f\|_{b,3}|y-z|.\end{aligned}$$

And for the third term we obtain:

$$\begin{aligned}|\mathcal{T}_3| &\le 2\vartheta^2\|f\|_{b,3}\left(\int_0^1\kappa\mathrm{d}\kappa\right)\int_\Lambda |h(s,z,\lambda)|\cdot|h(s,y,\lambda)-h(s,z,\lambda)|\Pi(\mathrm{d}\lambda)\\ &\le D_2\vartheta^2\|f\|_{b,3}|y-z|.\end{aligned}$$

Collecting all three terms, we get that $\forall s\ge 0$

$$|\psi_s(y)-\psi_s(z)| \le |\mathcal{T}_1|+|\mathcal{T}_2|+|\mathcal{T}_3| \le |y-z|(2+\sqrt{2})D_2\vartheta^2\|f\|_{b,3}, \tag{61}$$

and, in view of (53) and (58),

$$\|\psi_s\|_L \le (4+\sqrt{2})D_2\vartheta^2\|f\|_{b,3} < \infty, \tag{62}$$

where the constant on the right-hand side doesn't depend on $s$. Therefore, the limit (52) is proved, namely, $\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$,

$$E\left\{\int_0^t \int_\Lambda \left\langle \omega_{s,k} - \omega_s,\, f(\cdot + \vartheta h(s,\cdot,\lambda)) - f - \vartheta h(s,\cdot,\lambda)\cdot\nabla f\right\rangle \Pi(\mathrm{d}\lambda)\mathrm{d}s\right\}^2 \longrightarrow 0 \text{ as } k \to \infty. \tag{63}$$

Let us prove similar results for each term in the equation (3). $\forall y \in \mathbb{R}^2$, $\forall s \geq 0$, let

$$\widehat{\psi}_{s,y}(x) := \int_\Lambda (f(x + \vartheta h(s,x,\lambda)) - f(x))\,(f(y + \vartheta h(s,y,\lambda)) - f(y))\,\Pi(\mathrm{d}\lambda). \tag{64}$$

Then

$$\begin{aligned}
&E\left(\int_0^{t+}\int_\Lambda \left\langle \omega_{s-,k} - \omega_{s-},\, f(\cdot + \vartheta h(s,\cdot,\lambda)) - f\right\rangle \widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda)\right)^2 \\
&= E\int_0^t \int_\Lambda \left\langle \omega_{s,k} - \omega_s,\, f(\cdot + \vartheta h(s,\cdot,\lambda)) - f\right\rangle^2 \Pi(\mathrm{d}\lambda)\mathrm{d}s \\
&= \int_0^t E\int_{\mathbb{R}^2} \langle \omega_{s,k} - \omega_s, \widehat{\psi}_{s,y}\rangle(\omega_{s,k} - \omega_s)(\mathrm{d}y)\mathrm{d}s \\
&\leq \int_0^t E\int_{\mathbb{R}^2} \left|\langle \omega_{s,k} - \omega_s, \widehat{\psi}_{s,y}\rangle\right| |\omega_{s,k} - \omega_s|(\mathrm{d}y)\mathrm{d}s \\
&\leq (a^+ + a^-)\int_0^t E \sup_{y\in\mathbb{R}^2} \left|\langle \omega_{s,k} - \omega_s, \widehat{\psi}_{s,y}\rangle\right| \mathrm{d}s.
\end{aligned}$$

Let's show that $\forall y \in \mathbb{R}^2$, $\forall s \geq 0$, $f \in C_b^3(\mathbb{R}^2, \mathbb{R})$,

$$\|\widehat{\psi}_{s,y}(\cdot)\|_L \leq \text{const} < \infty, \tag{65}$$

where the above constant doesn't depend on $s$ and $y$. Note that

$$\begin{aligned}
\left|\widehat{\psi}_{s,y}(x)\right| &\leq \int_\Lambda \vartheta^2 \left|h(s,x,\lambda)\cdot\nabla f|_{x+\tau\vartheta h(s,x,\lambda)}\right| \left|h(s,y,\lambda)\cdot\nabla f|_{y+\tau'\vartheta h(s,y,\lambda)}\right| \Pi(\mathrm{d}\lambda) \\
&\leq 2\vartheta^2 \max_{i=1,2}\sup_{z\in\mathbb{R}^2} |\partial_i f(z)|^2 \int_\Lambda |h(s,x,\lambda)|\,|h(s,y,\lambda)|\,\Pi(\mathrm{d}\lambda) \\
&\leq 2\vartheta^2 \|f\|_{b,3}^2 \left(\int_\Lambda |h(s,x,\lambda)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} \left(\int_\Lambda |h(s,y,\lambda)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} \leq 2D_2\vartheta^2\|f\|_{b,3}^2.
\end{aligned} \tag{66}$$

On the other hand,

$$\begin{aligned}
&\left|\widehat{\psi}_{s,y}(x) - \widehat{\psi}_{s,y}(z)\right| \\
&\leq \sqrt{2}\vartheta\|f\|_{b,3}\int_\Lambda |h(s,y,\lambda)|\,|f(x + \vartheta h(s,x,\lambda)) - f(x) - f(z + \vartheta h(s,z,\lambda)) + f(z)|\,\Pi(\mathrm{d}\lambda) \\
&\leq \sqrt{2}\vartheta\|f\|_{b,3}\left(\int_\Lambda |h(s,y,\lambda)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_\Lambda |f(x + \vartheta h(s,x,\lambda)) - f(x) - f(z + \vartheta h(s,z,\lambda)) + f(z)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} \\
&\leq \sqrt{2D_2}\vartheta\|f\|_{b,3}\left(\int_\Lambda |f(x + \vartheta h(s,x,\lambda)) - f(x) - f(z + \vartheta h(s,z,\lambda)) + f(z)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}}.
\end{aligned}$$

$\forall x, z \in \mathbb{R}^2$, $\forall s \geq 0$, using the integral form of the remainder, we have

$$f(z+\vartheta h(s,z,\lambda)) = f(x+\vartheta h(s,x,\lambda)) + (z-x+\vartheta[h(s,z,\lambda)-h(s,x,\lambda)]) \cdot \int_0^1 (\nabla f)|_{x+\vartheta h(s,x,\lambda)+\kappa(z-x)+\kappa\vartheta(h(s,z,\lambda)-h(s,x,\lambda))}\mathrm{d}\kappa,$$

$$f(z) = f(x) + (z-x)\cdot\int_0^1 (\nabla f)|_{x+\kappa(z-x)}\mathrm{d}\kappa.$$

Therefore,

$$\begin{aligned} &f(x+\vartheta h(s,x,\lambda)) - f(x) - f(z+\vartheta h(s,z,\lambda)) + f(z) \\ &\quad = (z-x)\cdot\int_0^1 (\beta(s,x,z,\lambda,\kappa)\cdot\nabla)(\nabla f)\,|_{x+\kappa(z-x)+\tau\beta(s,x,z,\lambda,\kappa)}\mathrm{d}\kappa \\ &\qquad + \vartheta[h(s,x,\lambda)-h(s,z,\lambda)]\cdot\int_0^1 (\nabla f)|_{x+\kappa(z-x)+\beta(s,x,z,\lambda,\kappa)}\mathrm{d}\kappa, \end{aligned} \tag{67}$$

for some $0 \le \tau \le 1$ and where for notational convenience we defined the function:

$$\beta(s,x,z,\lambda,\kappa) := \kappa\vartheta h(s,z,\lambda) + (1-\kappa)\vartheta h(s,x,\lambda). \tag{68}$$

Then

$$\begin{aligned} &|f(x+\vartheta h(s,x,\lambda)) - f(x) - f(z+\vartheta h(s,z,\lambda)) + f(z)| \\ &\quad \le 2|z-x|\|f\|_{b,3}\int_0^1 |\beta(s,x,z,\lambda,\kappa)|\,\mathrm{d}\kappa + \vartheta\,\|f\|_{b,3}\,|h(s,x,\lambda)-h(s,z,\lambda)|. \end{aligned}$$

Therefore, by Minkowski's inequality,

$$\begin{aligned} &\left(\int_\Lambda |f(x+\vartheta h(s,x,\lambda)) - f(x) - f(z+\vartheta h(s,z,\lambda)) + f(z)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} \\ &\quad \le 2|z-x|\|f\|_{b,3}\left(\int_\Lambda\left\{\int_0^1 |\beta(s,x,z,\lambda,\kappa)|\,\mathrm{d}\kappa\right\}^2\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} \\ &\qquad + \vartheta\,\|f\|_{b,3}\left(\int_\Lambda |h(s,x,\lambda)-h(s,z,\lambda)|^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} \\ &\quad \le 2|z-x|\|f\|_{b,3}\left(\int_\Lambda\left\{\int_0^1 |\kappa\vartheta h(s,z,\lambda)+(1-\kappa)\vartheta h(s,x,\lambda)|\,\mathrm{d}\kappa\right\}^2\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} + \vartheta\,\|f\|_{b,3}\sqrt{D_2}|z-x| \\ &\quad \le 2|z-x|\|f\|_{b,3}\left(\int_\Lambda\left\{\vartheta|h(s,z,\lambda)|\int_0^1\kappa\mathrm{d}\kappa + \vartheta|h(s,x,\lambda)|\int_0^1(1-\kappa)\mathrm{d}\kappa\right\}^2\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} \\ &\qquad + \vartheta\,\|f\|_{b,3}\sqrt{D_2}|z-x| \\ &\quad \le \vartheta|z-x|\|f\|_{b,3}\left(\int_\Lambda\{|h(s,z,\lambda)|+|h(s,x,\lambda)|\}^2\,\Pi(\mathrm{d}\lambda)\right)^{\frac{1}{2}} + \vartheta\,\|f\|_{b,3}\sqrt{D_2}|z-x| \\ &\quad \le \vartheta|z-x|\|f\|_{b,3}\left[\left\{\int_\Lambda |h(s,z,\lambda)|^2\Pi(\mathrm{d}\lambda)\right\}^{\frac{1}{2}} + \left\{\int_\Lambda |h(s,x,\lambda)|^2\Pi(\mathrm{d}\lambda)\right\}^{\frac{1}{2}}\right] \\ &\qquad + \vartheta\,\|f\|_{b,3}\sqrt{D_2}|z-x| \le 3\sqrt{D_2}\vartheta\|f\|_{b,3}|z-x|. \end{aligned}$$

Thus,

$$\left|\widehat{\psi}_{s,y}(x) - \widehat{\psi}_{s,y}(z)\right| \le 3\sqrt{2}\vartheta^2 D_2\|f\|_{b,3}^2|z-x|. \tag{69}$$

Therefore, $\forall s \geq 0, \forall y \in \mathbb{R}^2$,

$$\begin{aligned}\|\widehat{\psi}_{s,y}(\cdot)\|_L &\leq \sup_{|x-z|\geq 1} \left|\widehat{\psi}_{s,y}(x) - \widehat{\psi}_{s,y}(z)\right| + \sup_{|x-z|\leq 1, x\neq z} \frac{\left|\widehat{\psi}_{s,y}(x) - \widehat{\psi}_{s,y}(z)\right|}{|x-z|} \\ &\leq 2\vartheta^2 D_2 \|f\|_{b,3} + 3\sqrt{2}\vartheta^2 D_2 \|f\|_{b,3}^2 = \text{const} < \infty, \end{aligned} \tag{70}$$

where the above constant is independent of the choice of $y$ and $s$. Thus,

$$\sup_{y\in\mathbb{R}^2} \left|\langle \omega_{s,k} - \omega_s, \widehat{\psi}_{s,y}\rangle\right| \leq \text{const} \sup_{y\in\mathbb{R}^2} \left|\left\langle \omega_{s,k} - \omega_s, \frac{\widehat{\psi}_{s,y}}{\|\widehat{\psi}_{s,y}\|_L}\right\rangle\right|. \tag{71}$$

Therefore, we conclude that

$$\begin{aligned} &E\left(\int_0^{t+}\int_\Lambda \left\langle \omega_{s-,k} - \omega_{s-}, f(\cdot + \vartheta h(s,\cdot,\lambda)) - f\right\rangle \widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda)\right)^2 \\ &\leq (a^+ + a^-)\int_0^t E \sup_{y\in\mathbb{R}^2} \left|\langle \omega_{s,k} - \omega_s, \widehat{\psi}_{s,y}\rangle\right| \mathrm{d}s \\ &\leq \text{const}(a^+ + a^-)\int_0^t \left(E \sup_{y\in\mathbb{R}^2} \left\langle \omega_{s,k} - \omega_s, \frac{\widehat{\psi}_{s,y}}{\|\widehat{\psi}_{s,y}\|_L}\right\rangle^2\right)^{\frac{1}{2}} \mathrm{d}s \longrightarrow 0 \text{ as } k \to \infty, \end{aligned} \tag{72}$$

by (50) and the dominated convergence theorem. As a result we have that $\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$,

$$E\left(\int_0^{t+}\int_\Lambda \left\langle \omega_{s-,k} - \omega_{s-}, f(\cdot + \vartheta h(s,\cdot,\lambda)) - f\right\rangle \widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda)\right)^2 \longrightarrow 0 \text{ as } k \to \infty. \tag{73}$$

Similarly, for $j = 1, 2$,

$$\begin{aligned} &E\left(\int_0^t \int_{\mathbb{R}^2} \langle \omega_{s,k} - \omega_s, \Gamma(\cdot, v)\partial_j f\rangle W^j(\mathrm{d}s\mathrm{d}v)\right)^2 = E\int_0^t \int_{\mathbb{R}^2} \langle \omega_{s,k} - \omega_s, \Gamma(\cdot, v)\partial_j f\rangle^2 \mathrm{d}s\mathrm{d}v \\ &= \int_0^t E \iint \left(\int_{\mathbb{R}^2} \Gamma(x,v)\Gamma(y,v)\mathrm{d}v\right) \partial_j f(x)\partial_j f(y)(\omega_{s,k} - \omega_s)(\mathrm{d}x)(\omega_{s,k} - \omega_s)(\mathrm{d}y)\mathrm{d}s \\ &= \int_0^t (E\int \partial_j f(y) \left\langle \omega_{s,k} - \omega_s, \varphi_{s,y}^j\right\rangle (\omega_{s,k} - \omega_s)(\mathrm{d}y))\mathrm{d}s, \end{aligned}$$

where we set

$$\varphi_{s,y}^j(x) := \partial_j f(x) \int_{\mathbb{R}^2} \Gamma(x,v)\Gamma(y,v)\mathrm{d}v. \tag{74}$$

Then it is easy to see that $\forall s \geq 0, \forall y \in \mathbb{R}^2$, $\|\varphi_{s,y}^j(\cdot)\|_L \leq const < \infty$, where the constant on the right-hand side is independent of $s$ and $y$. Therefore,

$$\begin{aligned} &E\left(\int_0^t \int_{\mathbb{R}^2} \langle \omega_{s,k} - \omega_s, \Gamma(\cdot, v)\partial_j f\rangle W^j(\mathrm{d}s\mathrm{d}v)\right)^2 \\ &\leq \int_0^t E\int \sup_{z\in\mathbb{R}^2} |\partial_j f(z)||\langle \omega_{s,k} - \omega_s, \varphi_{s,y}^j\rangle||\omega_{s,k} - \omega_s|(\mathrm{d}y)\mathrm{d}s \\ &\leq \|f\|_{b,3}\int_0^t E\int \|\varphi_{s,y}^j\|_L \left|\left\langle \omega_{s,k} - \omega_s, \frac{\varphi_{s,y}^j}{\|\varphi_{s,y}^j\|_L}\right\rangle\right| |\omega_{s,k} - \omega_s|(\mathrm{d}y)\mathrm{d}s \\ &\leq \text{const}\ (a^+ + a^-)\|f\|_{b,3}\int_0^t E \sup_{y\in\mathbb{R}^2} \left|\left\langle \omega_{s,k} - \omega_s, \frac{\varphi_{s,y}^j}{\|\varphi_{s,y}^j\|_L}\right\rangle\right| \mathrm{d}s \end{aligned}$$

$$\leq \text{const}\ (a^+ + a^-)\|f\|_{b,3} \int_0^t \left( E \sup_{y\in\mathbb{R}^2} \left\langle \omega_{s,k} - \omega_s, \frac{\varphi^j_{s,y}}{\|\varphi^j_{s,y}\|_L} \right\rangle^2 \right)^{\frac{1}{2}} \mathrm{d}s \longrightarrow 0,$$

as $k \to \infty$, by (50) and the dominated convergence theorem.

Therefore, $\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$, $j = 1, 2$,

$$E\left( \int_0^t \int_{\mathbb{R}^2} \left\langle \omega_{s,k} - \omega_s, \Gamma(\cdot, v)\partial_j f \right\rangle W^j(\mathrm{d}s\mathrm{d}v) \right)^2 \longrightarrow 0 \text{ as } k \to \infty. \tag{75}$$

Also, $\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$, we have $\|\Delta f\|_L \leq c < \infty$, for some constant $c$, so that

$$\begin{aligned} E\left( \int_0^t \left\langle \omega_{s,k} - \omega_s, \nu\Delta f \right\rangle \mathrm{d}s \right)^2 &\leq t \int_0^t E\left\langle \omega_{s,k} - \omega_s, \nu\Delta f \right\rangle^2 \mathrm{d}s \\ &\leq T\nu^2 c^2 \int_0^t E \sup_{\|\varphi\|_L \leq 1} \left\langle \omega_{s,k} - \omega_s, \varphi \right\rangle^2 \mathrm{d}s \longrightarrow 0 \text{ as } k \to \infty. \end{aligned} \tag{76}$$

Moreover, by the choice of $K_\delta(\cdot)$ we have that $\forall s \geq 0$,

$$\|(U_\delta \omega_s) \cdot \nabla f\|_L \leq \text{const} < \infty, \tag{77}$$

for some constant *const*. Thus, by the dominated convergence theorem and (50),

$$\begin{aligned} E\left( \int_0^t \left\langle \omega_{s,k} - \omega_s, (U_\delta\omega_s) \cdot \nabla f \right\rangle \mathrm{d}s \right)^2 &\leq t \int_0^t E\left\langle \omega_{s,k} - \omega_s, (U_\delta\omega_s) \cdot \nabla f \right\rangle^2 \mathrm{d}s \\ &\leq T\ \text{const}^2 \int_0^t E\left\langle \omega_{s,k} - \omega_s, \frac{(U_\delta\omega_s)\cdot\nabla f}{\|(U_\delta\omega_s)\cdot\nabla f\|_L} \right\rangle^2 \mathrm{d}s \longrightarrow 0 \text{ as } k \to \infty. \end{aligned} \tag{78}$$

Finally it is easy to show that

$$E\left( \int_0^t \left\langle \omega_{s,k}, (U_\delta\omega_s - U_\delta\omega_{s,k}) \cdot \nabla f \right\rangle \mathrm{d}s \right)^2 \leq t \int_0^t E\left\langle \omega_s - \omega_{s,k}, \phi_s \right\rangle^2 \mathrm{d}s, \tag{79}$$

where

$$\phi_s(y) := \int K_\delta(x - y) \cdot \nabla f(x) \omega_{s,k}(\mathrm{d}x) \tag{80}$$

and, $\forall s \geq 0$, $\|\phi_s\|_L \leq \text{const} < \infty$, where the constant doesn't depend on $s$ and $k$.

Thus, by the dominated convergence theorem and (50),

$$E\left( \int_0^t \left\langle \omega_{s,k}, (U_\delta\omega_s - U_\delta\omega_{s,k}) \cdot \nabla f \right\rangle \mathrm{d}s \right)^2 \leq T\ \text{const}^2 \int_0^t E\left\langle \omega_s - \omega_{s,k}, \frac{\phi_s}{\|\phi_s\|_L} \right\rangle^2 \mathrm{d}s \longrightarrow 0 \text{ as } k \to \infty. \tag{81}$$

$\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$ let

$$\begin{aligned} F_t[f] := \langle \omega_t, f\rangle - \langle \omega_0, f\rangle &- \int_0^t \langle \omega_s, (U_\delta\omega_s)\cdot\nabla f + \nu\Delta f\rangle\, \mathrm{d}s \\ &- \int_0^t \int_\Lambda \langle \omega_s, f(\cdot + \vartheta h(s, \cdot, \lambda)) - f - \vartheta h(s, \cdot, \lambda)\cdot\nabla f\rangle\, \Pi(\mathrm{d}\lambda)\mathrm{d}s \\ &- \sqrt{2\nu} \sum_{j=1}^2 \int_0^t \int_{\mathbb{R}^2} \left\langle \omega_s, \Gamma(\cdot, v)\partial_j f \right\rangle W^j(\mathrm{d}s\mathrm{d}v) - \int_0^{t+} \int_\Lambda \langle \omega_{s-}, f(\cdot + \vartheta h(s, \cdot, \lambda)) - f\rangle\, \widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda). \end{aligned} \tag{82}$$

Define $F_{t,k}[f]$ in the same way as $F_t[f]$ but with $\omega_{t,k}$ in place of $\omega_t$. Note that $F_{t,k}[f] \equiv 0$, since $\forall k \geq 1$ $(\omega_{t,k})_{t\in[0,T]}$ is the solution to the Eq. (3) with the initial condition $\omega_{0,k} \in L_2(\Omega, \mathcal{M}_p(a^+, a^-))$, where we took $\gamma(\omega_{0,k}, \omega_0) \to 0$

as $k \to \infty$ a.s. Then, by (50), (63), (73), (75), (76), (78) and (81), $\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$ and for all continuity points $t \in [0, T]$ of $\omega_t$ (thus, almost everywhere on $[0, T]$),

$$
\begin{aligned}
E(F_t[f])^2 &= E(F_t[f] - F_{t,k}[f])^2 \\
&\le 8 \Bigg\{ E\left\langle \omega_t - \omega_{t,k},\, f\right\rangle^2 + E\left\langle \omega_0 - \omega_{0,k},\, f\right\rangle^2 \\
&+ E\left( \int_0^t \left\langle \omega_s - \omega_{s,k},\, (U_\delta \omega_s)\cdot \nabla f + \nu \Delta f \right\rangle \mathrm{d}s \right)^2 \\
&+ E\left( \int_0^t \left\langle \omega_{s,k},\, (U_\delta \omega_s - U_\delta \omega_{s,k})\cdot \nabla f \right\rangle \mathrm{d}s \right)^2 \\
&+ E\left( \int_0^t \int_\Lambda \left\langle \omega_s - \omega_{s,k},\, f(\cdot + \vartheta h(s,\cdot,\lambda)) - f - \vartheta h(s,\cdot,\lambda)\cdot \nabla f \right\rangle \Pi(\mathrm{d}\lambda)\mathrm{d}s \right)^2 \\
&+ 2\nu \sum_{j=1}^{2} E\left( \int_0^t \int_{\mathbb{R}^2} \left\langle \omega_s - \omega_{s,k},\, \Gamma(\cdot, v)\partial_j f \right\rangle W^j(\mathrm{d}s\mathrm{d}v) \right)^2 \\
&+ E\left( \int_0^{t+} \int_\Lambda \left\langle \omega_{s-} - \omega_{s-,k},\, f(\cdot + \vartheta h(s,\cdot,\lambda)) - f \right\rangle \widetilde{\mathcal{N}}(\mathrm{d}s\mathrm{d}\lambda) \right)^2 \Bigg\} \longrightarrow 0,
\end{aligned}
$$

as $k \to \infty$. Since the left-hand side of the above inequality doesn't depend on $k$, then $\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$ and all continuity points $t \in [0, T]$ of $F_t[f]$, we have $E(F_t[f])^2 = 0$, which in turn implies that $F_t[f] = 0$ (a.s.). Since $(F_t[f])_{t\in[0,T]}$ is cadlag, then for all $f \in C_b^3(\mathbb{R}^2, \mathbb{R})$,

$$F_\cdot[f] = 0 \quad \text{a.s.}, \tag{83}$$

thus, $(\omega_t)_{t\in[0,T]}$ is a strong (in the stochastic analysis sense) solution of the vorticity model (3). □

**Diffusion approximation to the stochastic vorticity model**. Let $\Pi^n$ be a sequence of $\sigma$-finite measures on the measurable space $(\Lambda, \mathcal{E})$. Let $\mathcal{N}^n$, $n = 1, 2, \cdots$, be Poisson random measures on $\mathbb{R}^+ \times \Lambda$, independent of $W$, with characteristic measures $\Pi^n$ on $\Lambda$; $\widetilde{\mathcal{N}}^n(t, B) = \mathcal{N}^n(t, B) - t\Pi^n(B)$ are the corresponding compensated Poisson measures for $n = 1, 2, \cdots$. Denote by $L_2(\Omega; D([0, T], \mathcal{M}(b)))$ the space of square integrable $\mathcal{M}(b)$-valued cadlag stochastic processes. Fix an arbitrary collection of nonzero constants $\{a_i\}_{i=1}^N$ such that $\sum_{a_i>0} a_i = a^+$ and $\sum_{a_i<0} a_i = -a^-$ and $a^+ - a^- = b$. Let $\{\xi^{(i),n}\}_{i=1}^N$ $(n = 1, 2, \ldots)$ be $\mathbb{R}^2$-valued $\mathcal{F}_0$-adapted random variables, such that

$$\sup_n \sum_{i=1}^{N} E\left|\xi^{(i),n}\right|^2 < \infty, \tag{84}$$

and independent of the $\sigma$-fields generated by given $W$ and $\{\mathcal{N}^n\}_{n\ge1}$. Let $\{h^n(t, y, \lambda)\}_{n\ge1}$ be a sequence of measurable functions from $[0, T] \times \mathbb{R}^2 \times \Lambda$ to $\mathbb{R}^2$. Moreover, assume that $\forall t \in [0, T]$, the map $y \in \mathbb{R}^2 \mapsto h^n(t, y, \cdot) \in L_2(\Lambda, \Pi^n; \mathbb{R}^2)$ is continuous and the continuity is uniform in $n$. Let us also assume that there exists a positive constant $D_2$ (independent of $n$) such that:

$$\int_\Lambda \left|h^n(t, y_1, \lambda) - h^n(t, y_2, \lambda)\right|^2 \Pi^n(\mathrm{d}\lambda) \le D_2|y_1 - y_2|^2 \quad \text{for } \forall y_1, y_2 \in \mathbb{R}^2, \forall t \in [0, T], \forall n \ge 1, \tag{85}$$

and

$$\int_\Lambda |h^n(t, y, \lambda)|^2 \Pi^n(\mathrm{d}\lambda) \le D_2 \quad \text{for } y \in \mathbb{R}^2, \forall t \ge 0, \forall n \ge 1. \tag{86}$$

**Theorem 3.** *For each $n \in \mathbb{N}$, let $(\omega_t^{N,n})_{t\in[0,T]}$ be the empirical signed measure-valued process, as in Lemma 2, associated with the evolution of $N$ point vortices given by (11) and (12) but where $h$ is replaced with $h^n$ and $\widetilde{\mathcal{N}}$ is*

*replaced with $\widetilde{\mathcal{N}}^n$, and the initial positions are taken to be $\{\xi^{(i),n}\}_{i=1}^N$. For each $n \geq 1$, $(\omega_t^{N,n})_{t\in[0,T]}$ is a solution of the following nonlinear jump-diffusion model: $\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$, $\forall t \in (0, T)$,*

$$\mathrm{d}\langle \omega_t, f\rangle = \langle \omega_t, \nu\Delta f + (U_\delta \omega_t)\cdot\nabla f\rangle \mathrm{d}t + \int_\Lambda \langle \omega_t, f(\cdot + \vartheta h^n(t,\cdot,\lambda)) - f - \vartheta h^n(t,\cdot,\lambda)\cdot\nabla f\rangle \Pi^n(\mathrm{d}\lambda)\mathrm{d}t$$
$$+ \sqrt{2\nu}\sum_{j=1}^2 \int_{\mathbb{R}^2} \langle \omega_t, \Gamma(\cdot, v)\partial_j f\rangle W_j(\mathrm{d}t\mathrm{d}v) + \int_\Lambda \langle \omega_{t-}, f(\cdot + \vartheta h^n(t,\cdot,\lambda)) - f\rangle \widetilde{\mathcal{N}}^n(\mathrm{d}t\mathrm{d}\lambda), \tag{87}$$

*with the initial condition: $\omega_t|_{t=0} = \sum_{i=1}^N a_i \delta_{\xi^{(i),n}}$. Assume that there exists a measurable map $H : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $\forall t \in [0, T]$, $c > 0$ and compact subset $C_0 \subset \mathbb{R}^2$ we have*

$$\lim_{n\to\infty} \sup_{y\in C_0} \Pi^n\left\{\lambda \in \Lambda : \left|h^n(t, y, \lambda)\right| > c\right\} = 0, \tag{88}$$

$$\lim_{n\to\infty} \sup_{y_1, y_2 \in C_0} \left\| \int_\Lambda h^n(t, y_1, \lambda)\left(h^n(t, y_2, \lambda)\right)' \Pi^n(\mathrm{d}\lambda) - H(t, y_1)(H(t, y_2))' \right\|_{2\times 2} = 0, \tag{89}$$

$$\left(\text{where norm } \|\cdot\|_{2\times 2} \text{ is defined as } \left\|\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}\right\|_{2\times 2} := \sum_{i,j=1}^2 |c_{ij}|\right)$$

*and*

$$\lim_{M\to\infty} \sup_{y\in C_0, n\in\mathbb{N}} \int_\Lambda \left|h^n(t, y, \lambda)\right|^2 I_{\{|h^n(t,y,\lambda)|\geq M\}} \Pi^n(\mathrm{d}\lambda) = 0. \tag{90}$$

*Moreover, suppose $\xi^{(i),n} \xrightarrow{\mathcal{L}} \xi^{(i)}$ as $n \to \infty$ for each $i = 1, \ldots, N$, where $\{\xi^{(i)}\}_{i=1}^N$ are $\mathcal{F}_0$-measurable square-integrable random variables independent of $W$. (Here '$\xrightarrow{\mathcal{L}}$' denotes convergence in distribution). Then*

$$\omega^{N,n} \xrightarrow{\mathcal{L}} \mu^N \text{ as } n \to \infty,$$

*where $\mu^N = (\mu_t^N)_{t\in[0,T]} \in L_2(\Omega; C([0, T], \mathcal{M}(b)))$ is a solution of the following diffusion type SPDE: $\forall f \in C_b^3(\mathbb{R}^2, \mathbb{R})$, $\forall t \in (0, T)$,*

$$\mathrm{d}\langle \mu_t, f\rangle = \langle \mu_t, (U_\delta \mu_t)\cdot\nabla f + \nu\Delta f\rangle \mathrm{d}t + \left\langle \mu_t, \frac{\vartheta^2}{2}\sum_{k,l=1}^2 (H(t,\cdot)(H(t,\cdot))')_{kl}\partial_{kl}^2 f\right\rangle \mathrm{d}t$$
$$+ \sqrt{2\nu}\sum_{j=1}^2 \int_{\mathbb{R}^2} \left\langle \mu_t, \Gamma(\cdot, v)\partial_j f\right\rangle W^j(\mathrm{d}t\mathrm{d}v) + \vartheta \left\langle \mu_t, H(t,\cdot)\cdot\nabla f\right\rangle \mathrm{d}B(t) \tag{91}$$

*with the initial condition:*

$$\mu_t|_{t=0} = \sum_{i=1}^N a_i \delta_{\xi^{(i)}}, \tag{92}$$

*where $B = (B(t))_{t\geq 0}$ is a one-dimensional standard Brownian motion, independent of the Brownian sheet $W$ and $\{\xi^{(i)}\}_{i=1}^N$.*

**Proof.** For $n \geq 1$, let $X^n \equiv \left(x_1^n(t), \ldots, x_N^n(t)\right)_{t\in[0,T]} \in L_2(\Omega; D([0, T], \mathbb{R}^{2N}))$ be the unique strong solution of the stochastic system: $\forall t \geq 0$, $i = 1, \ldots, N$,

$$x^{(i),n}(t) = \xi^{(i),n} + \int_0^t \sum_{j=1}^N a_j K_\delta(x^{(i),n}(s) - x^{(j),n}(s))\mathrm{d}s$$
$$+ \sqrt{2\nu}\int_0^t \int_{\mathbb{R}^2} \mathbf{\Gamma}(x^{(i),n}(s), v)W(\mathrm{d}s\mathrm{d}v) + \vartheta \int_0^{t+}\int_\Lambda h^n(s, x^{(i),n}(s-), \lambda)\widetilde{\mathcal{N}}^n(\mathrm{d}s\mathrm{d}\lambda). \tag{93}$$

Then, by Theorem 8.3.3 in [12], $X^n \xrightarrow{\mathcal{L}} Y$ as $n \to \infty$, where $Y \equiv (y^{(1)}(t), \ldots, y^{(N)}(t))_{t\in[0,T]}$ is the unique strong solution of the stochastic system:

$$y^{(i)}(t) = \xi^{(i)} + \int_0^t \sum_{j=1}^N a_j K_\delta(y^{(i)}(s) - y^{(j)}(s))\mathrm{d}s + \sqrt{2\nu}\int_0^t \int_{\mathbb{R}^2} \Gamma(y^{(i)}(s), v)W(\mathrm{d}s\mathrm{d}v)$$
$$+ \vartheta \int_0^t H(s, y^{(i)}(s))\mathrm{d}B(s), \quad t \geq 0,\ i = 1, \ldots, N. \tag{94}$$

Then, by Itô's formula, $\mu_t^N(\mathrm{d}x) := \sum_{i=1}^N a_i \delta_{y^{(i)}(t)}(\mathrm{d}x)$ is a solution to (91) and (92). Define a metric

$$\rho_V(\mu, \mu^*) := \inf_{\theta\in\Theta}\left\{\sup_{0\leq t\leq T} V(\mu_t, \mu^*_{\theta(t)}) + \beta(\theta)\right\}, \quad \forall \mu, \mu^* \in D([0, T], \mathcal{M}(b)).$$

Let $\Phi^N$ be the map from $(D([0, T], \mathbb{R}^{2N}), \rho_N)$ into $(D([0, T], \mathcal{M}(b)), \rho_V)$, defined by:

$$\Phi^N(x) = \left(\sum_{i=1}^N a_i \delta_{x^{(i)}(t)}\right)_{t\in[0,T]}, \quad \forall x = (x^{(1)}(t), \ldots, x^{(N)}(t)) \in D([0, T], (\mathbb{R}^2)^N).$$

It suffices to show that $\Phi^N$ is continuous, since the latter, together with $X^n \xrightarrow{\mathcal{L}} Y$, implies that $\Phi^N(X^n) \xrightarrow{\mathcal{L}} \Phi^N(Y)$ by the continuous mapping theorem.

$\forall x = (x^{(1)}(t), \ldots, x^{(N)}(t))_{t\in[0,T]}$, $y = (y^{(1)}(t), \ldots, y^{(N)}(t))_{t\in[0,T]} \in D([0, T], \mathbb{R}^{2N})$,

$$\rho_V(\Phi^N(x), \Phi^N(y)) = \inf_{\theta\in\Theta}\left\{\sup_{0\leq t\leq T} V\left(\sum_{i=1}^N a_i \delta_{x^{(i)}(t)}, \sum_{i=1}^N a_i \delta_{y^{(i)}(\theta(t))}\right) + \beta(\theta)\right\},$$

where

$$V\left(\sum_{i=1}^N a_i \delta_{x^{(i)}(t)}, \sum_{i=1}^N a_i \delta_{y^{(i)}(\theta(t))}\right) = \sup_{\|\varphi\|_L\leq 1}\left|\sum_{i=1}^N a_i\left[\varphi(x^{(i)}(t)) - \varphi(y^{(i)}(\theta(t)))\right]\right|$$
$$\leq \sup_{\|\varphi\|_L\leq 1}\sum_{i=1}^N |a_i|\left|\varphi(x^{(i)}(t)) - \varphi(y^{(i)}(\theta(t)))\right| \leq \max(a^+, a^-)\sum_{i=1}^N |x^{(i)}(t) - y^{(i)}(\theta(t))|.$$

Thus,

$$\rho_V\left(\Phi^N(x), \Phi^N(y)\right) \leq \inf_{\theta\in\Theta}\left\{\sup_{0\leq t\leq T}\max(a^+, a^-)\left(\sum_{i=1}^N |x^{(i)}(t) - y^{(i)}(\theta(t))|\right) + \beta(\theta)\right\}$$
$$\leq \max(a^+, a^-, 1)\rho_N(x, y).$$

And we obtain that $\Phi^N$ is a continuous map from the metric space $(D([0, T], \mathbb{R}^{2N}), \rho_N)$ into the metric space $(D([0, T], \mathcal{M}(b)), \rho_V)$. Therefore, the required result follows. □

## 5. Conclusions

In this work we provided the background for stochastic partial differential equation modeling for the vorticity of a two-dimensional incompressible homogeneous viscous flow. Specifically we showed how to construct a class of solutions to the vorticity SPDE (which is a nonlinear measure-valued jump-diffusion) arising from the interacting particle system perspective and obtained a continuous diffusion approximation to it by controlling the magnitude and frequency of jumps.

Direct comparison of our SPDE (3) with the (regularized) Navier–Stokes model (10) reveals that, in the absence of jumps ($\theta = 0$) and under the appropriate conditions on the stochastic terms in the equation, the model provides a natural stochastic approximation to the classical (deterministic) Navier–Stokes model. On the other hand, if one were to explore the impact of jump-discontinuities in the context of mean-field limit results, the resulting deterministic

evolution equation will have the form of the classical Navier–Stokes equation but with an additional "correction" term. Since the classical Navier–Stokes equation loses its predictive power for large Reynolds numbers, the idea of having extra "correction" terms in the deterministic Navier–Stokes model appears well worthy of further study. Finally stochastic vorticity equation provides a natural alternative to the Navier–Stokes equation (in its velocity form) perturbed by external random forces and, being partly a product of numerical point vortex methods, is expected to be more computationally tractable.

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